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Throughout this document, K is a field and $n \geq 0$ is an integer. We write

$$\mathbb{P}^n(K) = (K^{n+1} \setminus \{0\})/K^\times,$$

and denote the class of a nonzero vector $a = (a_0, \ldots, a_n)$ by $[a] = (a_0 : \cdots : a_n)$.

Exercise 1.

Let K^{sep} be a separable closure of K, and let

$$\mathrm{Aut}(K^{\mathrm{sep}}/K) = \{\sigma \in \mathrm{Aut}(K^{\mathrm{sep}}) : \sigma|_K = \mathrm{id}_K\}.$$

(a) Prove that

$$K = \{a \in K^{\mathrm{sep}} : \sigma(a) = a \text{ for every } \sigma \in \mathrm{Aut}(K^{\mathrm{sep}}/K)\}.$$

(b) Take $K = \mathbb{R}$ and $K^{\mathrm{sep}} = \mathbb{C}$. Determine $\mathrm{Aut}(\mathbb{C}/\mathbb{R})$ and verify the equality above directly in this case.

Exercise 2.

Let $K \subseteq L$ be a finite field extension. Show that the inclusion $K^{n+1} \subseteq L^{n+1}$ induces a well-defined injective map

$$\begin{aligned} \mathbb{P}^n(K) &\longrightarrow \mathbb{P}^n(L), \\ (a_0 : \cdots : a_n) &\longmapsto (a_0 : \cdots : a_n). \end{aligned}$$

Exercise 3.

Let $K \subseteq L$ be a field extension and $\sigma \in \mathrm{Aut}(L/K)$.

(a) Show that the automorphism

$$\begin{aligned} L^{n+1} &\xrightarrow{\sigma} L^{n+1} \\ (a_0, \ldots, a_n) &\longmapsto (\sigma(a_0), \ldots, \sigma(a_n)) \end{aligned}$$

induces a bijection

$$\begin{aligned} \mathbb{P}^n(L) &\longrightarrow \mathbb{P}^n(L), \\ (a_0 : \cdots : a_n) &\longmapsto (\sigma(a_0) : \cdots : \sigma(a_n)), \end{aligned}$$

whose restriction to $\mathbb{P}^n(K)$ is the identity.

(b) Take $K = \mathbb{R}$, $L = \mathbb{C}$, and let σ be complex conjugation. Compute $\sigma(P)$ and $\sigma(Q)$ for

$$P = (1 : i : 0), \qquad Q = (i : 2i : 0),$$

and determine which of these projective points are fixed by σ.

Exercise 4.

For a group H acting on a set X, write $X^H = \{x \in X : h(x) = x \text{ for every } h \in H\}$.

(a) If $K \subseteq L$ is a finite Galois extension, prove that

$$\mathbb{P}^n(L)^{\mathrm{Aut}(L/K)} = \mathbb{P}^n(K).$$

(b) If K^{sep} is a separable closure of K, prove that

$$\mathbb{P}^n(K^{\mathrm{sep}})^{\mathrm{Aut}(K^{\mathrm{sep}}/K)} = \mathbb{P}^n(K).$$

(c) Let $L = \mathbb{Q}(\sqrt[3]{2}) \subseteq \mathbb{R}$. Determine $\mathrm{Aut}(L/\mathbb{Q})$, and show that

$$\mathbb{P}^1(L)^{\mathrm{Aut}(L/\mathbb{Q})} \neq \mathbb{P}^1(\mathbb{Q}).$$

Why does this not contradict (a)?

Exercise 5.

Let $d \geq 0$ be an integer.

(a) Let $F \in K[x_0, \dots, x_n]$ be homogeneous of degree d. Prove that

$$F(\lambda a_0, \dots, \lambda a_n) = \lambda^d F(a_0, \dots, a_n)$$

for every $\lambda \in K^\times$ and every $(a_0, \dots, a_n) \in K^{n+1}$.

(b) Assume that K is infinite. Prove that a polynomial $F \in K[x_0, \dots, x_n]$ is homogeneous of degree d if and only if

$$F(\lambda a_0, \dots, \lambda a_n) = \lambda^d F(a_0, \dots, a_n)$$

for every $\lambda \in K^\times$ and every $(a_0, \dots, a_n) \in K^{n+1}$.

(c) What if K is finite ?

Exercise 6.

If $F \in K[x_0, \dots, x_n]$ is homogeneous and L/K is a field extension, its zero locus over L

$$F(L) := \{(a_0 : \cdots : a_n) \in \mathbb{P}^n(L) : F(a_0, \dots, a_n) = 0\}.$$

(a) Show that

$$\sigma(F(L)) = F(L) \subseteq \mathbb{P}^n(L) \qquad \text{for every } \sigma \in \mathrm{Aut}(L/K).$$

(b) If $K \subseteq L$ is a finite Galois extension, prove that

$$F(L)^{\mathrm{Aut}(L/K)} = F(K).$$

(c) If K^{sep} is a separable closure of K, prove that

$$F(K^{\mathrm{sep}})^{\mathrm{Aut}(K^{\mathrm{sep}}/K)} = F(K).$$

Exercise 7.

For $d \geq 0$, let

$$K[x_0, \dots, x_n]_d = \{F \in K[x_0, \dots, x_n] \mid F \text{ is homogeneous of degree } d\}.$$

Prove the following formulas:

(a)

$$\dim_K K[x, y, z]_d = \binom{d+2}{d}.$$

(b)

$$\dim_K K[x_0, \dots, x_n]_d = \binom{d+n}{d}.$$

Exercise 8.

(a) Let $\varphi \in \mathrm{Aut}(K^{n+1})$. Show that the map

$$\begin{aligned} \widetilde{\varphi} : \mathbb{P}^n(K) &\longrightarrow \mathbb{P}^n(K), \\ [a] &\longmapsto [\varphi(a)] \end{aligned}$$

is well-defined and bijective.

(b) Let $\varphi, \psi \in \mathrm{Aut}(K^{n+1})$. Prove that

$$\widetilde{\varphi} = \widetilde{\psi} \quad \Longleftrightarrow \quad \text{there exists } \lambda \in K^\times \text{ such that } \varphi = \lambda\psi.$$

Exercise 9.

Given $(a, b, c) \in K^3 \setminus \{(0, 0, 0)\}$, find a matrix $A \in \mathrm{GL}_3(K)$ such that

$$\begin{pmatrix} a & b & c \end{pmatrix} A = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$