# 23 October

Throughout, $K$ is a field. For a prime $p$, we denote by $\mathbb{F}_p$ the field with $p$ elements ($\mathbb{Z}/p\mathbb{Z}$).

## Exercise 1.

(a) Show that the number of points in $\mathbb{P}^2(\mathbb{F}_q)$ is $1 + q + q^2$.

(b) Deduce the number of lines in $\mathbb{P}^2(\mathbb{F}_q)$.

## Exercise 2.

Let $P = (a : b : c) \in \mathbb{P}^2(K)$.

(a) show that there exists $\psi \in \mathrm{PGL}_3(K)$ such that

$$\psi(P) = (0 : 0 : 1).$$

Is the choice of $\psi$ unique ?

(b) We denote by $\varphi_0$ the affine chart centered at $(0 : 0 : 1)$. Recall that $\varphi_P := \varphi_0 \circ \psi$ is an affine chart centered at $P$. Choose linear forms $\psi_x, \psi_y, \psi_z$ in $K[x, y, z]$ such that $\psi = (\psi_x : \psi_y : \psi_z)$. Let $f \in K[x, y]$. For every $d \geq \deg f$, show that

$$\psi_z^d f(\varphi_P(x, y, z))$$

is a homogeneous polynomial of degree $d$ in $K[x, y, z]$.

(c) Let $F \in K[x, y, z]$ be a nonzero homogeneous polynomial of degree $d \geq 1$. Show that $F(\varphi_P^{-1}(x, y)) \in K[x, y]$ has degree at most $d$. If $P \in F(K)$, why is it nonconstant ?

(d) Write $F(\varphi_P^{-1}(x, y)) = f_0 + f_1 + \cdots + f_d$, $f_i \in K[x, y]$ homogeneous of degree $i$. Show that

$$f_0 = 0 \quad \Longleftrightarrow \quad P \in F(K).$$

(e) Assume that $P$ is nonsingular. Show that the zero locus of $\psi_z f_1(\varphi_P(x, y, z))$, the tangent line at $P$, is the unique line in $\mathbb{P}^2(K)$ that contains $\varphi_P^{-1}(\{(\alpha, \beta) \in K^2 : f_1(\alpha, \beta) = 0\})$.

## Exercise 3.

Over an arbitrary field $K$, let $F(x, y, z) = xy - z^2$.

(a) Describe the points of $F(K)$ with $z \neq 0$ in affine coordinates and find all points of $F(K)$ with $z = 0$.

(b) Show that

$$\nu : \mathbb{P}^1(K) \longrightarrow F(K), \qquad (s : t) \longmapsto (s^2 : t^2 : st)$$

is well-defined and bijective. Give an explicit inverse (you can treat $x = 0$ separately).

(c) Deduce $\#F(\mathbb{F}_q)$.

## Exercise 4.

Let $F \in K[x, y, z]$ be a nonzero homogeneous polynomial of positive degree, and let $P = (a : b : c) \in F(K)$. Prove that $P$ is nonsingular for $F$ if and only if

$$\left(\frac{\partial F}{\partial x}(a, b, c), \frac{\partial F}{\partial y}(a, b, c), \frac{\partial F}{\partial z}(a, b, c)\right) \neq (0, 0, 0).$$

### Exercise 5.

Consider the following affine plane curves in $\mathbb{A}^2(\mathbb{R})$ defined by:

$$\begin{aligned} A: &\quad y^2 - x^3 = 0, \\ B: &\quad y^2 - x^3 - x^2 = 0, \\ C: &\quad (x^2+y^2)^2 + 3x^2y - y^3 = 0, \\ D: &\quad (x^2+y^2)^3 - 4x^2y^2 = 0. \end{aligned}$$

Show that $P = (0,0)$ is the only singular point of each curve.

### Exercise 6.

Find the singular points and the tangent lines at the singular points, for each of the following affine curves in $\mathbb{A}^2(\mathbb{C})$:

(a) $y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy = 0$.

(b) $x^4 + y^4 - x^2y^2 = 0$.

(c) $x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1 = 0$.

(d) $y^2 + (x^2-5)(4x^4 - 20x^2 + 25) = 0$.

Sketch the real locus of the curve in part (d), that is, its intersection with $\mathbb{A}^2(\mathbb{R}) \subset \mathbb{A}^2(\mathbb{C})$.

### Exercise 7.

For each of the following projective plane curves over $\mathbb{C}$, find their singular points, the multiplicities and the tangent lines at those points.

(a) $xy^4 + yz^4 + xz^4 = 0$.

(b) $x^2y^3 + x^2z^3 + y^2z^3 = 0$.

(c) $y^2z - x(x-z)(x-\lambda z) = 0$, where $\lambda \in \mathbb{C}$.

(d) $x^n + y^n + z^n = 0$, where $n > 0$.

### Exercise 8.

Write $\infty = (1:0)$, $0 = (0:1)$, and $1 = (1:1)$ in $\mathbb{P}^1(K)$.

(a) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K)$, describe its action in the affine coordinate $t = x/y$. What is the image of $\infty$?

(b) Let $P, Q, R$ be three distinct points of $\mathbb{P}^1(K)$. We want to prove that there is a unique element of $\mathrm{PGL}_2(K)$ sending $P, Q, R$ to $\infty, 0, 1$, respectively.

  (a) Choose nonzero representatives $p, q \in K^2$ of $P, Q$. Explain why $(p, q)$ is a basis of $K^2$. Write a nonzero representative of $R$ as $r = \alpha p + \beta q$. Explain why $\alpha \neq 0$ and $\beta \neq 0$.

  (b) Show that there is a unique invertible linear map $A : K^2 \to K^2$ satisfying

$$A(p) = \begin{pmatrix} \alpha^{-1} \\ 0 \end{pmatrix}, \qquad A(q) = \begin{pmatrix} 0 \\ \beta^{-1} \end{pmatrix}.$$

  (c) Show that the only element of $\mathrm{PGL}_2(K)$ fixing $\infty, 0, 1$ is the identity. Deduce that $[A]$ is the unique element of $\mathrm{PGL}_2(K)$ sending $P, Q, R$ to $\infty, 0, 1$, respectively.

(c) Deduce $\#\,\mathrm{PGL}_2(\mathbb{F}_q)$.