30 October

Throughout Exercises 1 and 2, let K be a field of characteristic different from 2, and consider the conic

$$C: \quad X^2 + Y^2 - Z^2 = 0 \qquad \text{in } \mathbb{P}^2_K.$$

Write $P = (-1 : 0 : 1) \in C(K)$.

Exercise 1.

For each $t \in K$, let L_t be the line through P and $(0 : t : 1)$.

(a) Show that L_t has equation $Y = t(X + Z)$.

(b) Using the substitution $U = X + Z$, find the intersections of L_t and C and the multiplicities.

(c) Show that the line $X + Z = 0$ meets C twice at P. What is this line for P? Justify.

Exercise 2.

Define

$$\nu : \mathbb{P}^1(K) \longrightarrow C(K), \qquad (s : t) \longmapsto (s^2 - t^2 : 2st : s^2 + t^2).$$

(a) Prove that ν is well-defined and that its image lies on C.

(b) Show that $\nu(0 : 1) = P$. If $R = (X : Y : Z) \in C(K)$ and $R \neq P$, show that $X + Z \neq 0$ and that

$$\nu^{-1}(R) = (X + Z : Y).$$

Deduce that ν is a bijection.

(c) Check that $\nu(1 : t)$ is the intersection of L_t and C (of Exercice 1) other than P.

Exercise 3.

Work over $\mathbb{Q}$, and consider

$$D: \quad F(X, Y, Z) = 2YZ - 3X^2 - 4XZ - 5Z^2 = 0.$$

(a) Check that $P_0 = (0 : 1 : 0)$ lies on D. Find its tangent line.

(b) Make the linear change of projective coordinates

$$X' = X, \qquad Y' = \frac{2Y - 4X - 5Z}{3}, \qquad Z' = Z.$$

Show that it is invertible and that $F = 3(Y'Z' - (X')^2)$.

(c) Set $U = Y' + Z'$, $V = Y' - Z'$, and $W = 2X'$. Show that this change is invertible and that

$$\frac{4}{3}F = U^2 - V^2 - W^2.$$

(d) How is D related to C of Exercice 1?

For the Exercices 4, 5, 6, and 7, k is a field of characteristic different from 2.

Exercise 4.

Consider the conic in $\mathbb{P}^2(k)$ defined by

$$F = uv - y^2.$$

In the affine chart $v \neq 0$, the equation of the conic is $f = u - y^2$.

(a) Show that

$$\varphi : k[u, y]/(f) \longrightarrow k[t], \qquad y \longmapsto t, \quad u \longmapsto t^2$$

is an isomorphism of rings. Deduce that every point of $F(k)$ with $v \neq 0$ can be written uniquely as $(t^2 : t : 1)$. Find the point of $F(k)$ with $v = 0$.

(b) For a linear form $l \in k[u, y, v]$ defining a line in $\mathbb{P}^2(k)$, we use $\varphi(l)$ as shorthand for $\varphi([l(u, y, 1)])$. Let

$$l(k) \cap F(k) = \{(a_1 : b_1 : 1), (a_2 : b_2 : 1)\},$$

where the two points are distinct. Show that there is a $\lambda \in k^\times$ such that

$$\varphi(l) = \lambda(t - b_1)(t - b_2).$$

(c) Suppose that

$$l(k) \cap F(k) = \{(a : b : 1), (1 : 0 : 0)\}.$$

Show that there is a $\lambda \in k^\times$ such that

$$\varphi(l) = \lambda(t - b).$$

Exercise 5.

(a) Let E/k be a separable extension of degree 2, and let $\theta \in E \setminus k$ have minimal polynomial $T^2 + pT + q \in k[T]$. Show that the other root is $-p - \theta \in E$. Why does it follow that E/k is Galois?

(b) Now let $a \in k^\times$ be a nonsquare and write $E = k(\sqrt{a})$, with $\theta = \sqrt{a}$. Show that the nonidentity k-automorphism σ of E sends θ to $-\theta$.

(c) For $u = x + y\theta \in E$ define its norm by $\mathrm{Norm}_{E/k}(u) = u\,\sigma(u)$. Compute the norm and show that $\mathrm{Norm}_{E/k}(uv) = \mathrm{Norm}_{E/k}(u)\mathrm{Norm}_{E/k}(v)$.

Exercise 6.

Let $a, b \in k^\times$ and let (a, b) be the quaterion algebra with basis $1, i, j, ij$, with $i^2 = a$, $j^2 = b$, and $ij = -ji$.

(a) For $q = x + yi + zj + wij$ define $\bar{q} = x - yi - zj - wij$. Check, using the basis and $q\bar{q} = \bar{q}q$, that $\overline{qr} = \bar{r}\,\bar{q}$ for all $q, r \in (a, b)$.

(b) Deduce that $N(qr) = N(q)N(r)$ for all $q, r \in (a, b)$, where $N(q) = q\bar{q} \in k$.

Exercise 7.

Assume that $a \in k^\times$ is a nonsquare and $b \in k^\times$. Choose $x, y \in k$ such that

$$c := x^2 - ay^2 = \mathrm{Norm}_{k(\sqrt{a})/k}(x + y\sqrt{a}) \neq 0.$$

Inside (a, b), put $J = (x + yi)j = xj + yij$.

(a) Compute J^2 and $iJ + Ji$.

(b) Express J and iJ in the basis (j, ij). Show that the resulting 2×2 coefficient matrix has determinant c.

(c) Deduce that $(1, i, J, iJ)$ is a basis of (a, b) and that

$$(a, b) \cong (a, bc).$$

(d) Show that $1 - a$ and $-a$ are norms of nonzero elements of $k(\sqrt{a})$. Deduce that

$$(a, 1 - a) \cong (a, 1) \quad \text{and} \quad (a, -a) \cong (a, 1)$$

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(e) Find matrices $I, J \in M_2(k)$ with $I^2 = a\,\mathrm{Id}$, $J^2 = \mathrm{Id}$, and $IJ = -JI$. Show that Id, I, J, IJ form a basis of $M_2(k)$. Deduce that both quaternion algebras in part (d) are isomorphic to $M_2(k)$.