06&09 November

Throughout, let k be a field of characteristic different from 2, and let $a, b \in k^\times$. Write (a, b) for the quaternion algebra with basis $(1, i, j, ij)$ and relations

$$i^2 = a, \qquad j^2 = b, \qquad ij = -ji.$$

We write $C_{a,b} \subset \mathbb{P}^2_k$ for the conic

$$ax^2 + by^2 = z^2.$$

Exercise 1.

Let $\sigma : (a, b) \to (a, b)$ be a k-linear map such that $\sigma(1) = 1$ and $\sigma(q)q \in k$ for every $q \in (a, b)$.

(a) Show that $\sigma(q) + q \in k$ for all $q \in (a, b)$.

(b) For $q \in \{i, j, ij\}$, check that $\sigma(q) = \bar{q}$.

(c) Deduce that $\sigma(q) = \bar{q}$ for evey $q \in (a, b)$.

Exercise 2.

(a) Let $b \in k^\times$ and set

$$1 = I_2, \qquad i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad j = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}, \qquad ij = \begin{pmatrix} 0 & b \\ -1 & 0 \end{pmatrix}.$$

as a basis of the quaterion algebra $M_{2\times 2}(k) = (1, b)$. Given $A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$, write A uniquely in the form $x1 + yi + zj + wij$. Deduce that

$$\bar{A} = \begin{pmatrix} u & -s \\ -t & r \end{pmatrix}.$$

(b) Deduce that $N(A) = \det(A)$.

(c) Let Q be a split quaterion algebra. Show using Exercice 1 that it has a basis (e, f, g, h) in which

$$\mathrm{N}(xe + yf + zg + wh) = xy - zw.$$

(d) Conversely, if a quaternion algebra has such a basis, use e to show that it is split.

Exercise 3.

Let $k = \mathbb{F}_q = \mathbb{Z}/p\mathbb{Z}$ with q an odd prime number. You may use that $\mathbb{F}_q^\times$ is cyclic.

(a) How many elements of $\mathbb{F}_q$ are squares (including 0)?

(b) Show that the subsets $\{ax^2 : x \in \mathbb{F}_q\}$ and $\{1 - by^2 : y \in \mathbb{F}_q\}$ of $\mathbb{F}_q$ have a common element.

(c) Deduce that every quaternion algebra over $\mathbb{F}_q$ is split.

Exercise 4.

Let t be an indeterminate. Consider the quaternion algebra $(a,b) \otimes_k k(t)$ over $k(t)$.

(a) Explain in two ways why, if (a,b) is split over k, then $(a,b) \otimes_k k(t)$ is split over $k(t)$.

(b) For the converse, proceed as follows:

(i) Suppose $P \in C_{a,b}(k(t))$. Show that one can write

$$P = (X(t) : Y(t) : Z(t)),$$

where $X, Y, Z \in k[t]$ are not all divisible by t.

(ii) Deduce that if $(a,b) \otimes_k k(t)$ is split over $k(t)$, then (a,b) is split over k.

Exercise 5.

Consider the quaternion algebra (a,t) over $k(t)$. Its associated conic is

$$ax^2 + ty^2 = z^2.$$

(a) If a is a square in k, exhibit a $k(t)$-point of this conic. Deduce that (a,t) is split over $k(t)$.

(b) For the converse, proceed as follows:

(i) Suppose the conic has a point $P \in C_{a,t}(k(t))$. As in Exercise 4, write

$$P = (X(t) : Y(t) : Z(t)),$$

where $X, Y, Z \in k[t]$ are not all divisible by t. Show that $X(0)$ and $Z(0)$ cannot both be zero.

(ii) Deduce that a is a square in k.

Exercise 6.

(a) Compute $(Xi + Yj + Zij)^2$. Show that

$$C_{a,b} : \quad ax^2 + by^2 = z^2$$

is isomorphic to the conic

$$aX^2 + bY^2 - abZ^2 = 0$$

using an invertible change of coordinate.

(b) Let (c,d) have basis $(1, i', j', i'j')$. Suppose that

$$\phi : (a,b) \longrightarrow (c,d)$$

is an isomorphism of k-algebras. Write $\phi(i)$, $\phi(j)$ and $\phi(ij)$ in the basis $(1, i', j', i'j')$. Show that in each expression the coefficient of 1 is zero.

Hint: If $q = r + Xi' + Yj' + Zi'j'$ is not a scalar and $q^2 \in k$, compute the coefficient of $i', j', i'j'$ in q^2.

(c) The coefficients found in part (b) define a change of coordinates in $\mathbb{P}^2(k)$. Show that it sends the conic $aX^2 + bY^2 - abZ^2 = 0$ to the conic $cX^2 + dY^2 - cdZ^2 = 0$.

(d) Deduce that $C_{a,b}$ and $C_{c,d}$ are isomorphic over k.

Exercise 7.

Suppose the same quaternion algebra A has two bases $(1, i, j, ij)$ and $(1, I, J, IJ)$, with

$$i^2 = a, \quad j^2 = b, \quad ij = -ji, \qquad I^2 = c, \quad J^2 = d, \quad IJ = -JI.$$

Put $T = jJ - Jj$ and assume $T^2 \neq 0$.

(a) Show directly that $Tj = -jT$ and $TJ = -JT$. Write J in the first basis and show that T has the form $ui + vij$ for some $u, v \in k$. Deduce that $e := T^2 \in k^\times$.

(b) Show that $(1, T, j, Tj)$ and $(1, T, J, TJ)$ are both basis of A. *Hint:* For the first basis, express T, Tj in terms of i, ij and compute the 2×2 determinant. Treat the second basis similarly.

(c) Conclude that $(a, b) \cong (e, b) \cong (e, d) \cong (c, d)$.