November 15

Throughout, k is an algebraically closed field of characteristic different from 2 and 3. We use homogeneous coordinates $(x : y : z)$ on $\mathbb{P}^2(k)$, write $F(k)$ for the projective zero locus of a homogeneous polynomial F, and use $i(P, L, F)$ for the intersection multiplicity of a line L and F at P, as in the lecture.

Exercise 1.

Let $F \in k[x, y, z]$ be a conic and let L be a line that is not a component of F.

(a) Let $P \in F(k) \cap L(k)$. Write $f = F(\varphi_P^{-1}(x, y)) = f_0 + f_1 + f_2$, where f_i is homogeneous of degree i, and parametrize L by $t \mapsto (at, bt)$. Explain why $i(P, L, F)$ is either 1 or 2.

(b) If P is nonsingular, prove that $i(P, L, F) = 2$ precisely when L is the tangent line at P. If P is singular, show that $i(P, L, F) = 2$ for every line through P that is not a component of F.

(c) For $F = xy - z^2$, find all intersection points and their multiplicities for each of the lines $z = 0$, $x = 0$, and $x - y = 0$. Which of these lines are tangent to the conic?

Exercise 2.

Let $F \in k[x, y, z]$ be homogeneous of degree $d \geq 1$, and let L be a line that is not a component of F. Choose linearly independent vectors $p, q \in k^3$ representing two points on L.

(a) Show that $(s : t) \mapsto [sp + tq]$ parametrizes all points of $L(k)$, and that $H(s, t) := F(sp + tq)$ is a nonzero homogeneous polynomial of degree d.

(b) Factor H into linear factors over k. In a local coordinate on $\mathbb{P}^1$ at a zero $(s_0 : t_0)$, explain why the order of vanishing of H equals $i([s_0p + t_0q], L, F)$.

(c) Deduce that

$$\sum_{P \in F(k) \cap L(k)} i(P, L, F) = d.$$

Why is the assumption that L is not a component necessary?

Exercise 3.

Consider the cubic

$$F = y^2z - x(x - z)(x - 2z), \qquad C = F(k), \qquad O = (0 : 1 : 0).$$

(a) Show that O is the only point of C on $z = 0$. In the chart $y = 1$, compute $i(O, z, F)$. Show that O is nonsingular and identify its tangent line. Deduce that O is a flex point.

(b) Show that all affine points of C are nonsingular.

(c) Find the intersection of C with $y = 0$, including all multiplicities. Check your answer against Exercise 2(c).

Exercise 4.

For a Weierstrass equation

$$E_{A,B} : \quad y^2z - x^3 - Axz^2 - Bz^3 = 0,$$

use the lecture's invariants c_4, c_6, Δ, and $j = c_4^3/\Delta$.

(a) Derive from their definitions that

$$c_4 = -48A, \qquad c_6 = -864B, \qquad \Delta = -16(4A^3 + 27B^2), \qquad j = 1728\frac{4A^3}{4A^3 + 27B^2}.$$

When is $E_{A,B}$ nonsingular ?

(b) If $a \in k^\times$, compute j for $y^2z - x^3 - az^3$ and for $y^2z - x^3 - axz^2$. Show that both curves are nonsingular.

(c) Let $\alpha \in k \setminus \{0, 1728\}$ and put

$$t = \frac{27\alpha}{4(1728 - \alpha)}.$$

Using t, show that every $\alpha \in k$ occurs as the j-invariant of some nonsingular elliptic curve.

Exercise 5.

Let F be a nonsingular cubic:

$$F = y^2z + a_1xyz + a_3yz^2 - x^3 - a_2x^2z - a_4xz^2 - a_6z^3.$$

(a) For $u \in k^\times$, let $F_u(x, y, z) = u^6F(u^{-2}x, u^{-3}y, z)$. Find its new coefficients $a_1', a_2', a_3', a_4', a_6'$. Show that $j(F_u) = j(F)$.

(b) For $r, s, t \in k$, use the change of variables $(X, Y, Z) = (x + rz,\ y + sx + tz,\ z)$ into F.

(a) Find the new coefficients a_i's and check that $c_4' = c_4, c_6' = c_6, \Delta' = \Delta$.

(b) Conclude that j is unchanged by every admissible change of variables.

(c) Let $A, B, A', B' \in k$. Write $E_{A,B}$ and $E_{A',B'}$ for the projective curves defined, respectively, by

$$y^2z = x^3 + Axz^2 + Bz^3 \qquad \text{and} \qquad y^2z = x^3 + A'xz^2 + B'z^3.$$

Suppose that both are nonsingular and have the same j-invariant.

(a) If $j = 0$, show that $A = A' = 0$. If $j = 1728$, show that $B = B' = 0$. In each case, find $u \in k^\times$ such that $A' = u^4A$ and $B' = u^6B$.

(b) Suppose $j \notin \{0, 1728\}$. Show first that

$$A^3(B')^2 = (A')^3B^2.$$

Choose $u \in k^\times$ with $u^4 = A'/A$. Explain how to choose u so that also $u^6 = B'/B$.

(c) Deduce that $E_{A,B}$ is isomorphic to $E_{A',B'}$ by an admissible change of variables.

Exercise 6.

Consider

$$E:\ y^2z = x^3 + z^3, \qquad O = (0 : 1 : 0), \quad P = (0 : 1 : 1), \quad Q = (2 : 3 : 1).$$

For $A, B \in E(k)$. Define $A \oplus B := O \cdot (A \cdot B)$.

(a) Check that E is nonsingular, that its tangent at O is $z = 0$, and that $i(O, z, E) = 3$.

(b) Show that the line through P and Q is $y = x + z$. Find $P \cdot Q$, then compute $P \oplus Q$.

(c) Show that the line through O and an affine point $R = (a : b : 1)$ is $x = az$. Show that $O \oplus R = R$ and $R \oplus (a : -b : 1) = O$.