# Artin fans

**Definition** (fs log scheme)

A fs log scheme is a log scheme $X$, st every $x \in X$ admits an étale neighborhood $V$ and a morphism

$$V \longrightarrow \mathcal{U}_\sigma$$

to an affine toric variety, st the log structure $M_X|_V$ is the pull-back of the divisorial log structure on $\mathcal{U}_\sigma$ induced by the toric boundary $\partial \mathcal{U}_\sigma$.

(this is a local chart or local toric model)

How to find the local toric model?

Given $(X, M_X)$ a fs log scheme, $p \in X$, then $\overline{M}_{X,p}$ is a toric monoid (in particular $\operatorname{Spec} k[\overline{M}_{X,p}]$ is an affine toric variety).

**Example**

Let $(X, D = D_1 + \dots + D_k)$ an snc pair, $n := \dim X$.

let $p$ be a point in the loc. closed strata

$$\left(\bigcap_{i \in I} D_i\right) \setminus \left(\bigcup_{i \in \{1,\dots,k\} \setminus I} D_i\right)$$

with $I \subset \{1, \dots, k\}$

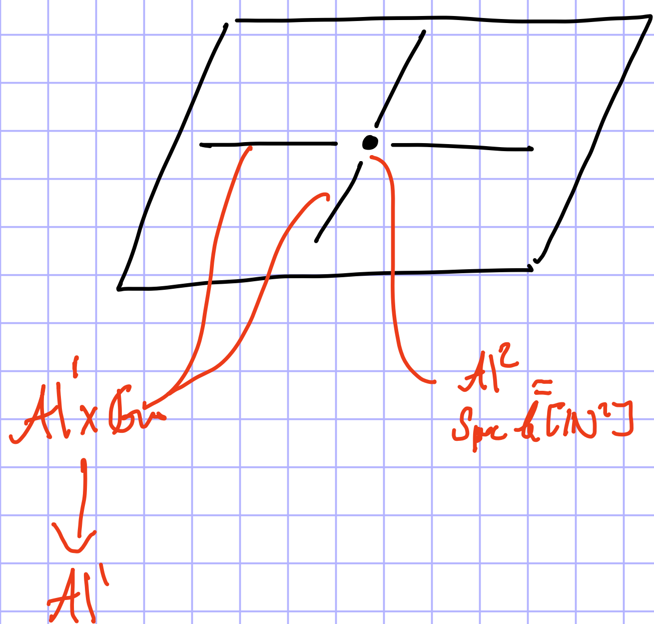
then there exists a (Zariski) neighborhood $V \subseteq X$ of $p$ with $V \simeq \mathbb{A}^{|I|} \times \mathbb{G}_m^{n-|I|}$

↳ affine toric var

where $D$ is identified with the toric boundary of $\mathbb{A}^{|I|} \times \mathbb{G}_m^{n-|I|}$

$(X, D)$ has a natural log structure: on $V$, it is pulled-back from the divisorial log structure of $\mathbb{A}^{|I|} \times \mathbb{G}_m^{n-|I|}$ through the $\simeq$.

(Note that for general fs log schemes, we don't ask for an "$\simeq$").




$\mathbb{A}^2$

$D = V(xy)$

- If $(X, M_X)$ is a fs log scheme let $V \to U_\sigma$ be a local toric model. For any $g \in T_\sigma$ the dense torus, with the action of $T_\sigma$ on $U_\sigma$, we get a different local toric model

$$V \to U_\sigma \xrightarrow{} U_\sigma$$

action of $g$ on $U_\sigma$

So toric models are not unique $\Rightarrow$ there is no canonical way of gluing these affine toric varieties into a toric variety $Z$ and obtain a global morphism $X \to Z$.

<u>Definition</u> (Artin Cone)

Given an affine toric variety $U_\sigma$ with dense torus $T_\sigma$, the corresponding Artin cone is the stack quotient

$$A_\sigma := [\, U_\sigma / T_\sigma \,]$$

Given $V \longrightarrow \mathcal{U}_\sigma$

post compose with the quotient morphism

$$V \to \mathcal{U}_\sigma \xrightarrow{q} \mathcal{A}_\sigma$$

if $V \to \mathcal{U}_\sigma \xrightarrow[T_\sigma \circlearrowright \text{action}]{} \mathcal{U}_\sigma \xrightarrow{q} \mathcal{A}_\sigma$

this gives then rise to a unique morphism

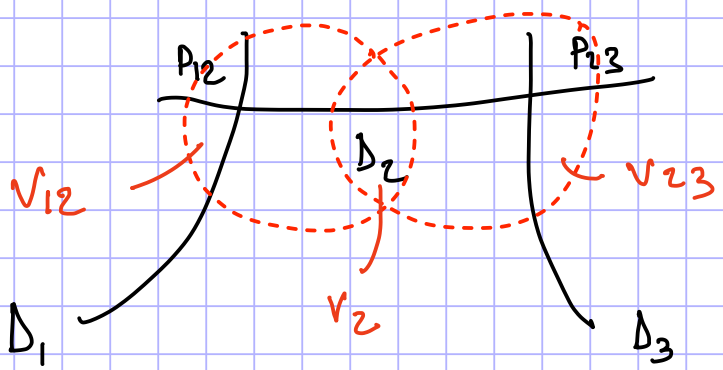
$$V \longrightarrow \mathcal{A}_\sigma$$

Remarks

- $\mathcal{A}_\sigma$ is not a toric variety but it inherits from $\mathcal{U}_\sigma$ its combinatorial aspects.
- The log structure on $\mathcal{U}_\sigma$ is $T_\sigma$-invariant $\rightsquigarrow$ it descends to a log structure on the quotient $\mathcal{A}_\sigma$: it is also the log structure induced by the boundary $\partial \mathcal{A}_\sigma = [\partial \mathcal{U}_\sigma / T_\sigma]$.
- Pulling back this log structure through $V \longrightarrow \mathcal{A}_\sigma$ produces the log structure $M_{X|V}$.

## Example

Consider a snc divisor of the form




Choose $V_{12}$ and $V_{23}$ around the closed points $P_{12}$ and $P_{23}$ so that $V_{ij}$ only intersects the loc. closed strata containing $P_{ij}$.

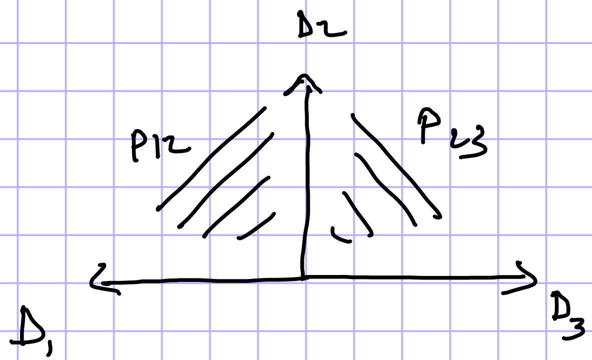
$V_2 := V_{12} \cap V_{23}$ is open around an interior point of $D_2$.

$$V_{12} \simeq \mathbb{A}^2 \times \mathbb{G}_m \longrightarrow [\mathbb{A}^2 / \mathbb{G}_m^2]$$

$$V_{23} \longrightarrow [\mathbb{A}^2 / \mathbb{G}_m^2]$$

$$V_2 \simeq \mathbb{A}^1 \times \mathbb{G}_m \longrightarrow [\mathbb{A}^1 \times \mathbb{G}_m / \mathbb{G}_m^2] = [\mathbb{A}^1 / \mathbb{G}_m]$$

The cone complex is




two copies of $\mathbb{R}^2_{\geq 0}$ glued together along the common face $\mathbb{R}_{\geq 0}$

Likewise, by gluing two copies of $[\mathbb{A}^2/\mathbb{G}_m^2]$ along $[\mathbb{A}^1/\mathbb{G}_m]$

$$[\mathbb{A}^2/\mathbb{G}_m^2] \hookleftarrow [\mathbb{A}^1/\mathbb{G}_m] = [\mathbb{A}^1 \times \mathbb{G}_m/\mathbb{G}_m^2] \underset{\substack{\text{dense}\\ \text{open}}}{\hookrightarrow} [\mathbb{A}^2/\mathbb{G}_m^2]$$

We obtain an Artin (alg.) stack which has a cover by Artin cones.

We refer to the result as the Artin fan $\mathcal{A}_X$ of $(X, M_X)$

the unique toric models $V \to \mathcal{A}_\sigma$ glue to give a unique global morphism.

$$X \longrightarrow \mathcal{A}_X$$

# Artin fans

(Olsson)

Denote by $\mathcal{L}og$ the category fibered in groupoids over the category of alg. stacks.

obj: fs log alg. stacks

arrows: strict morphisms of log alg. stacks

Thm (Olsson)

① $\mathcal{L}og$ is an algebraic stack.

② $\mathcal{L}og$ admits an étale rep. cover by Artin cones.

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Let $\underline{X}$ be an alg stack. A morphism of alg. stacks $\underline{X} \longrightarrow \mathcal{L}og$ amounts to endowing $\underline{X}$ with a log structure.

Conversely, a log algebraic stack $X$ comes with a morphism $\underline{X} \longrightarrow \mathcal{L}og$.

## Definition

- Let $X$ be a log alg. stack. It is an Artin fan if $\underline{X} \to \mathcal{L}og$ is representable and étale.
- A morphism of AF is a morphism of log alg. stacks
- Any Artin fan has a strict étale cover by Artin cones (Abramovich_Wise Lemma 2.31 ref3).

## Example

- An Artin cone is an Artin fan (Olsson 2)

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Given $S$ an alg. stack

$$S \to (Sch)$$

Let $M_S$ a fs log structure on $S \rightsquigarrow S^{log}$ (log stack)

$$S^{log} \to (Lsch.)_{ét}$$

this is an alg. stack (Kato)

$S = [U_\sigma / T_\sigma]$

$S^{log} = \mathcal{A}_\sigma$ is in fact equivalent to a functor.

If $U_\sigma = \mathrm{Spec}\, k[P]$

$$(LSch) \longrightarrow Sets$$

$$Y \longmapsto \mathrm{Hom}_{mon}(P, \Gamma(Y, \overline{M}_Y))$$

(A) $V = \mathrm{Spec}\, k[P]^{\text{toric var}}$, then $V$ represents the functor

$$Y \longmapsto \mathrm{Hom}_{mon}(P, \Gamma(Y, \mathcal{O}_Y))$$

$T = \mathrm{Spec}\, k[P^{gp}]$ represents the functor

$$Y \longmapsto \mathrm{Hom}_{mon}(P^{gp}, \Gamma(Y, \mathcal{O}_Y)) =$$

$$\mathrm{Hom}_{mond}(P, \Gamma(Y, \mathcal{O}_Y^\times))$$

The action of T on V comes from the action of $\mathcal{O}_Y^\times$ on $\mathcal{O}_Y$ which is not free (in general)
($0 \in \mathcal{O}_Y$ has $\mathcal{O}_Y^\times$ as a stabilizer)

(B) $V = \mathrm{Spec}\, k[P]$ as a toric var seen as a log scheme, it represents the functor

$$(LSch) \longrightarrow Sets$$

$$Y \longmapsto \mathrm{Hom}_{mon}(P, \Gamma(Y, M_Y))$$

T represents the functor

$$Y \longmapsto \mathrm{Hom}_{mon}(P, \Gamma(Y, \mathcal{O}_Y^\times))$$

The action of T on V as a log scheme comes from the action of $\mathcal{O}_Y^\times$ on $M_Y$ which is free as Y is fs!

So $\underset{\underset{\mathcal{A}_\sigma}{\parallel}}{[V/_T]}^{\log}$ is an alg. space.

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<u>Corollary</u> $\mathrm{Hom}(\mathcal{A}_N, \mathcal{A}_M) = \mathrm{Hom}(M, N)$

<u>Artin fan of a log (scheme / alg stack)</u>

<u>Definition</u>

Let X be a log scheme. It is said to be small with respect to $x \in X$ if the restriction

$$\Gamma(X, \overline{M}_X) \longrightarrow \overline{M}_{X,x}$$

is an isomorphism

and $\{y \in X \mid \overline{M}_{X,y} \simeq \overline{M}_{X,x}\}$ is connected

We say that X is small if X is small wrt some $x \in X$.

Let $N := \Gamma(X, \overline{M}_X)$

Then we define the Artin fan of $X$ to be $A_N$.

$$N \xrightarrow[\sim]{} \Gamma(X, \overline{M}_X)$$

$$\rightsquigarrow \qquad X \longrightarrow A_N$$

In addition, if $X$ is small $X \to A_N$ is strict and initial among all strict morphisms from $X$ to an Artin fan.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\text{strict}"'] & A_N \arrow[d, dashed, "\text{unique (+strict)}"] \\ & A \end{tikzcd}$$

In particular $X \to A_N$ is functorial with respect to strict morphisms:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\text{strict}"'] & A_N \arrow[d, dashed] \\ Y \arrow[r] & A_M \end{tikzcd}$$

$Y$ small

## Example

$X = \mathbb{A}^2$ is a small log scheme with respect to the origin $\rightsquigarrow \mathcal{A}_X = [\mathbb{A}^2 / \mathbb{G}_m^2]$

Point of $\mathcal{A}_X \underset{1-1}{\longleftrightarrow}$ Cones of $\Sigma_{\mathbb{A}^2}$

orbits:

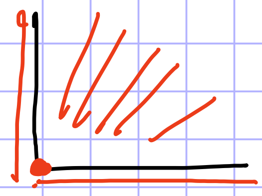
- origin
- axis − origin $\quad [\vec{0}/\mathbb{G}_m^2] \longleftrightarrow$
- torus

$\Sigma_X$

$\mathbb{R}^2_{\geq 0}$

$\mathbb{R}_{\geq 0} \times 0$

$0 \times \mathbb{R}_{\geq 0}$



$\{*\} = [\mathbb{G}_m^2/\mathbb{G}_m^2] \longleftrightarrow$ torus $\quad 0 \times 0$

$$\begin{tikzcd} B\mathbb{G}_m \bullet \arrow[r, squiggly] & \bullet\, B\mathbb{G}_m^2 = [\vec{0}/\mathbb{G}_m^2] \\ \{*\} \bullet \arrow[u, squiggly] \arrow[r, squiggly] & \bullet\, B\mathbb{G}_m \arrow[u, squiggly] \end{tikzcd}$$

$u \rightsquigarrow u'$ if $u' \in \overline{\{u\}}$

specialization maps

the orbit $\mathbb{G}_m / \mathbb{G}_m^2 = [B\mathbb{G}_m]$

**Theorem** (Prop 3.2.1 ref 2 : Abramovich – Chen – Marcus – Wise)

Let $X$ be a log alg. stack whose log strata are locally connected in the sm. top. Then there is an initial factorization map $X \to \mathcal{L}og$ through an strict étale morphism $\mathcal{A}_X \to \mathcal{L}og$ rep by alg spaces.

$$\begin{tikzcd} X \arrow[r, "\text{strict}"] \arrow[dr, "\text{strict}"'] & \mathcal{L}og \\ & \mathcal{A}_X \arrow[u, "\text{strict}"'] \end{tikzcd}$$

In particular, $X \to A_X$ is functorial wrt strict morphisms.

<u>Proof</u>:

1) Prove that every geom. point of $X$ has a smooth neigh which small.

2) $\{U_i \to X\}$ such a cover

refi of covers

$$\{V_i \xrightarrow[\text{strict}]{} V_j\}_{i,j \in \Delta}$$

(sm cover of $X$)

$$\begin{tikzcd}
A_{V_i} \arrow[r, "\varphi_{ij}"] \arrow[dr, "\text{s.ét}"'] & A_{V_j} \arrow[d, "\text{s.ét}"] \\
 & \mathrm{Log}
\end{tikzcd}$$

strict étale

Each $A_{V_i}$ represents an alg. space over $\mathrm{Log}$ $\rightsquigarrow$ an étale sheaf over $\mathrm{Log}$.

Take the colimit which is an étale sheaf over $\mathrm{Log}$ $\rightsquigarrow$ take its étale space: this is the Artin fan of $X$.

By the universal property of colimit we get the initial fact.

$\boxtimes$

## Counter eg:

$\mathbb{G}_m \circlearrowright \mathbb{A}^1 \quad x \longmapsto t^2 x$

$\hookrightarrow [\mathbb{A}^1 / \mathbb{G}_m]$ is an alg. stack but not
$\quad\;\; \mu_2$

an Artin fan.