§ 3.4 The Cotrace of Weil Differentials and the Hurwitz Genus Formula

$$\begin{array}{ccl} K & \subset & F \leftarrow \text{function field} \\ \cap & & \cap \leftarrow \text{finite separable extension} \\ K' & \subset & F' \end{array}$$

K': the c^{st} field of F'

K' / K: finite separable extension

Goal :

(1) To associate to each Weil differential of F/K a Weil differential of F'/K'.

(2) To Prove Hurwitz Genus Formula which provides a relation between the genus of F, the genus of F' and the different of F'/F.

Notations :

- $\mathbb{P}_F$ the set of places of F/K.
- If $P \in \mathbb{P}_F$, $\mathcal{O}_P$ the valuation ring of P
- $\mathcal{O}_P'$ the integral closure of $\mathcal{O}_P$ in F', ie $\mathcal{O}_P' = \bigcap\limits_{\substack{P'|P \\ P' \in \mathbb{P}_{F'}}} \mathcal{O}_{P'}$ $\qquad \mathcal{O}_P \subset \mathcal{O}_{P'}$, $\quad v_{P'}(x) = e \cdot v_P(x)$, $x \in F$
- $\alpha \in F'$, $Tr_{F'/F}(\alpha) := Tr(m_\alpha) \in F$, where

m_α is the F-linear map : $F' \longrightarrow F'$
$\qquad x \longmapsto \alpha x$.

Definition (Complementary module over $\mathcal{O}_P$)

$P \in \mathbb{P}_F$, the set

$$C_P := \{ z \in F' \mid Tr_{F'/F}(z \cdot \mathcal{O}_P') \subseteq \mathcal{O}_P \}$$

is called the complementary module over $\mathcal{O}_P$.

$\hookrightarrow$ it is an $\mathcal{O}_P'$-module by F-linearity of $Tr_{F'/F}$.

Proposition 3.4.2

(1) $\mathcal{O}_P' \subseteq C_P$.

(2) There exists $t \in F'$ (depending on P) st $C_P = t\,\mathcal{O}_P'$ (generated by t as $\mathcal{O}_P'$ module).
$\quad v_{P'}(t) \leq 0$ for all $P'|P$
and $\forall t' \in F'$ we have :
$$C_P = t'\,\mathcal{O}_P' \iff v_{P'}(t') = v_{P'}(t) \quad \text{for all } P'|P$$

(3) $C_P = \mathcal{O}_P'$ for almost all $P \in \mathbb{P}_F$.

Definition (different) :

Let $P \in \mathbb{P}_F$ and $C_P = t\,\mathcal{O}_P'$. We define the *different exponent* of P' over P for $P'|P$ by
$$d(P'|P) = -v_{P'}(t) \geq 0 \quad \leadsto \text{well defined by (b) of Prop 3.4.2}$$

Note that $d(P'|P) = 0$ for almost all $P \in \mathbb{P}_F$ and $P'|P$ by (3) of Prop 3.4.2

We define the different of F'/F the divisor

$$\mathrm{Diff}(F'/F) := \sum_{P\in \mathbb{P}_F} \sum_{P'|P} d(P'|P)\, P'$$

It is a divisor over F'.

Notations:

$\alpha : \mathbb{P}_F \to F$, $P \mapsto \alpha_P$ is an adele if $\alpha_P \in \mathcal{O}_P$ for almost all $P \in \mathbb{P}_F$
$\alpha = (\alpha_P)_{P\in \mathbb{P}_F}$

- A_F an adel space of F/K.
$A_F = \{\alpha \mid \alpha \text{ is adele of } F/K\}$ is a K-vs.
$\alpha \in A_F$, $v_P(\alpha) := v_P(\alpha_P)$, $P\in \mathbb{P}_F$ — α_P: P-Component of α

- For $A \in \mathrm{Div}(F)$, let
$A_F(A) = \{\alpha \in A_F \mid v_P(\alpha) \geq -v_P(A) \ \forall P \in \mathbb{P}_F\} \subseteq A_F$ (K-subspace)

- A Weil Differential ω of F/K is a K-linear map
$\omega : A_F \to K$
vanishing on $A_F(A) + F$ for some $A \in \mathrm{Div}(F)$.
- If $\omega \neq 0$ then we associate to it a divisor $(\omega) = \max\{A \in \mathrm{Div}(F) \mid \omega \text{ vanishes on } A_F(A) + F\}$

Definition: $A_{F'/F} := \{\alpha \in A_{F'} \mid \alpha_{P'} = \alpha_{Q'}, \text{ whenever } P'\cap F = Q'\cap F \text{ for } P', Q' \in \mathbb{P}_{F'}\} \subseteq A_{F'}$ (F'-subspace)
The trace $\mathrm{Tr}_{F'/F} : F' \to F$ can be extended to an F-linear map:

$$\mathrm{Tr}_{F'/F} : A_{F'/F} \longrightarrow A_F$$
$$(\mathrm{Tr}_{F'/F}(\alpha))_P := \mathrm{Tr}_{F'/F}(\alpha_{P'}) \quad \text{for } P'|P \quad (\text{well defined by})$$

- $(\mathrm{Tr}_{F'/F}(\alpha))_{P\in\mathbb{P}_F}$ is an adele of F/K:

If $\alpha \in A_{F'/F}$, $\alpha_{P'} \in \mathcal{O}_{P'}$ for almost all $P' \in \mathbb{P}_{F'}$ (because α is an adele in $A_{F'}$)
Moreover, $\alpha_{P'} \in \mathcal{O}_P' = \bigcap_{P'|P} \mathcal{O}_{P'}$, hence it is integral over $\mathcal{O}_P$

hence $\mathrm{Tr}_{F'/F}(\alpha_{P'}) \in \mathcal{O}_P$ for almost all $P \in \mathbb{P}_F$.

Theorem 3.4.6
(1) For every Weil differential ω of F/K, there exists a unique Weil differential ω' of F'/K' st

$$\mathrm{Tr}_{K'/K}(\underbrace{\omega'}_{\omega' : A_{F'} \to K'}(\alpha)) = \underbrace{\omega}_{\omega : A_F \to K}(\mathrm{Tr}_{F'/F}(\alpha)) \qquad (\text{uses } F'/F \text{ separable})$$

for all $\alpha \in A_{F'/F}$.
We call it the Cotrace of ω in F'/F, denoted $\mathrm{Cotr}_{F'/F}(\omega)$.

(2) If $\omega \neq 0$ and $(\omega) \in \mathrm{Div}(F)$ the divisor of ω, then

$$\mathrm{Cotr}_{F'/F}(\omega) = \mathrm{Con}_{F'/F}((\omega)) + \mathrm{Diff}(F'/F)$$

Theorem 3.4.13 (Hurwitz Genus formula)

Let F/K be an algebraic function field of genus g (Def 1.4.15), F'/F finite sep. Let K' the c^{st} field of F' and g' the genus of F'/K. Then

$$2g'-2 = \frac{[F':F]}{[K':K]}(2g-2) + \deg \mathrm{Diff}(F'/F)$$

Proof: Let $\omega \neq 0$ a Weil differential of F/K. By Theorem 3.4.6

$$(\mathrm{Cotr}_{F'/F}(\omega)) = \mathrm{Con}_{F'/F}((\omega)) + \mathrm{Diff}(F'/F)$$

$$\text{hence} \quad \deg(\underbrace{\mathrm{Cotr}_{F'/F}(\omega)}_{=(\omega')}) = \deg \mathrm{Con}_{F'/F}((\omega)) + \mathrm{Diff}(F'/F)$$

Cor 1.5.16, $\deg(\omega') = 2g'-2$

Cor 3.1.14, $\deg \mathrm{Con}_{F'/F}((\omega)) = \frac{[F':F]}{[K':K]}\deg(\omega)$

$$= \frac{[F':F]}{[K':K]}(2g-2)$$

Example if $F'/F = K(x)$ is finite separable, with cst field K ($K(x)$: rational field)

genus of $K(x)/K = 0$ by Prop 1.6.3

The formula gives, if g' is the genus of F'/K:

$$2g'-2 = -2[F':K(x)] + \deg \mathrm{Diff}(F'/K(x))$$

§ 3.5. The Different

Goal: Address the problem of computing the different, namely using the ramification index.

$$\begin{array}{ccc} K & \subset & F \\ \cap & & \cap \\ K' & \subset & F' \end{array}$$ function field ext. (F'/F finite separable)

K (hence K') is assumed perfect.

- Given $P \in \mathbb{P}_F$ and $P' \in \mathbb{P}_{F'}$ st $P' | P$, we relate $e(P'|P)$ (ramif index) and $d(P'|P)$ (diff exp)

Definition : F'/F an alg. extension of function fields, $P \in \mathbb{P}_F$.

(a) $P' \in \mathbb{P}_{F'}$ st $P'|P$ is said tamely (resp. widly) ramified if :
$\begin{cases} \bullet\ e(P'|P) > 1 \\ \bullet\ \mathrm{Char}\, K \nmid e(P'|P) \end{cases}$ $\{$ resp. $\mathrm{Char}(K) \mid e(P'|P)$.

(b) We say P is ramified (resp. unramified) in F'/F if there is at least one $P' \in \mathbb{P}_{F'}$ over P st $P'|P$ is ramified (resp. $P'|P$ is unramified for all $P'|P$). The place P is tamely ramified in F'/F if it is ramified in F'/F and no extension of P in F' is widely ramified. The place P is widely ramified in F'/F is there is at least one place $P'|P$ that is widely ramified.

(c) P is totally ramified in F'/F if there is only one extension $P' \in \mathbb{P}_{F'}$ st P'/P and $e(P'|P) = [F':F]$.

(d) F'/F is said to be ramified (resp. unr) if at least one $P \in \mathbb{P}_F$ is ramified in F'/F (resp. all $P \in \mathbb{P}_F$ is unramified).

(e) F'/F is said to be tame if no place $P \in \mathbb{P}_F$ is widely ramified in F'/F otherwise, widly ramified.

Theorem 3.5.1 (Dedekind's Different Theorem)

for $P'|P$:

(a) $d(P'|P) \geq e(P'|P) - 1$

(b) $d(P'|P) = e(P'|P) - 1$ iff $e(P'|P)$ is not divisible by char K. ← uses perfectness

In particular, if $\mathrm{Char}\, K = 0$, $d(P'|P) = e(P'|P) - 1$.

Corollary 3.5.6

Let F'/F finite sep. ext. of alg function fields having the same c^{st} field K. Let g (resp. g') be the genus of F/K (resp. F'/K) Then

$$2g' - 2 \geq [F':F](2g-2) + \sum_{P \in \mathbb{P}_F} \sum_{P'|P} (e(P'|P) - 1) \deg P'.$$

Equality hold iff $e(P'|P) - 1 = d(P'|P)$ for all $P'|P$

- So, for example, if F'/F is unramified, we are in (2) of the theorem so $\mathrm{Diff}(F'/F) = 0$ and
$$2g' - 2 = [F':F](2g-2)$$

- If F'/F is tamely ramified, we are also is (2) of the theorem and there is "=".
- If F'/F is widely ramified, we only get a lower bound via this formula.

Lemma 3.5.2 : Let F^*/F be an algebraic extension of function fields, $P \in \mathbb{P}_F$ and $P^* \in \mathbb{P}_{F^*}$ with $P^*|$

Let $\sigma \in Aut_F(F^*)$. Then:

(a) $\sigma(P^*) = \{\sigma(z) \mid z \in P^*\}$ is a place of F^*.

(b) $v_{\sigma(P^*)}(y) = v_{P^*}(\sigma^{-1}(y)) \quad \forall y \in F^*$.

(c) $\sigma(P^*) \mid P$.

(d) $e(\sigma(P^*)|P) = e(P^*|P)$.

Proof of Theorem 3.5.1 (a)

(A) It is enough to prove that $\underbrace{Tr_{F'/F}(t.\mathcal{O}_{P}') \subseteq \mathcal{O}_P}_{*}$ $\forall t \in F'$ st $\underbrace{v_{P'}(t) = 1 - e(P'|P)}_{**}$ $\forall P'|P$

$* \Rightarrow t \in C_P = t_0 . \mathcal{O}_P'$ for some $t_0 \in F'$

$\Rightarrow v_{P'}(t) = v_{P'}(t_0 z)$ for $z \in \mathcal{O}_P'$

$= v_{P'}(t_0) + v_{P'}(z)$

$= -d(P'|P) + v_{P'}(z) \geq -d(P'|P)$

By $**$ $d(P'|P) \geq e(P'|P) - 1$

(B) Let's show (A)

Let F^*/F be a finite Galois extension st

$F \subseteq F' \subseteq F^* \subseteq \Psi$

Since F'/F is separable, there are exactly $n \neq$ embeddings $\sigma_1, \dots, \sigma_n$ of F'/F into Ψ (an alg. closed field $\supseteq F'$), and for $x \in F$, we have } p82

$Tr_{F'/F}(x) = \sum_{i=1}^{n} \sigma_i(x)$ $\quad\hookrightarrow$ restrictions of embeddings $\sigma_1, \dots, \sigma_n$ of F^*/F into Ψ i.e. $\in Aut(F^*/F)$

hence, if $z \in \mathcal{O}_P'$:

$$Tr_{F'/F}(t.z) = \sum_{i=1}^{n} \sigma_i(t.z)$$

Fix $P^* \in \mathbb{P}_{F^*}$ st $P^*|P$.

Set $P_i^* := \underbrace{\sigma_i^{-1}(P^*)}_{\hookrightarrow \text{ a place in } F^*}$ and $P_i' := P_i^* \cap F'$

$$v_{P^*}(\sigma_i(t.z)) = v_{P^*}(\sigma_i(t)) + v_{P^*}(\sigma_i(z))$$

$\geq v_{P^*}(\sigma_i(t))$ $\quad \hookrightarrow z \in \mathcal{O}_P' \Rightarrow \sigma_i(z) \in \mathcal{O}_P'$

$= v_{P_i^*}(t)$ $\quad \Rightarrow v_{P^*}(\sigma_i(z)) \overset{4}{=} e(P^*|P')\, v_{P'}(\sigma_i(z))$ $\quad P' := P^* \cap F'$

(b)

$= e(P_i^*|P_i')\, v_{P_i'}(t)$

$= \underbrace{e(P_i^*|P_i')}_{>0 \text{ by def of } P_i'}(1 - e(P_i'|P))$

$**$

$> -e(P_i^*|P_i')\, e(P_i'|P) = -e(P_i^*|P)$

$= -e(P^*|P)$

(d)

Therefore $-e(P^*|P) < v_{P^*}(\sigma_i(t.z))$ $\quad \forall i$

$\Rightarrow < v_{P^*}(Tr_{F'/F}(t.z))$ $\quad \hookrightarrow \geq \min\{\sigma_i(t.z)\}$

$\Rightarrow -e(P^*|P) < e(P^*|P)\, v_P(Tr_{F'/F}(t.z))$

Finally $v_P(Tr_{F'/F}(t.z)) \geq 0 \Rightarrow Tr_{F'/F}(t.z) \in \mathcal{O}_P \Rightarrow$ (A) $\square$

Proposition 3.5.9 (Lüroth's Thm)

Every subfield of a rational function field is rational, i.e if $K \subsetneq F_0 \subseteq K(x)$ then $F_0 = K(y)$ for some $y \in F_0$.

Proof

1. $K(x)/F_0$ separable. Let $g_0 = \text{genus}(F_0/K)$

We prove that $\text{genus}(F_0/K) = 0$ and $\exists$ place of F_0 of $\deg = 1$ (prop 1.6.3)

the HG Formula, since $\text{genus}(K(x)/K) = 0$

$$-2 = \underbrace{[K(x):F_0]}_{\geq 0}(2g_0 - 2) + \underbrace{\deg \text{Diff}(K(x)/F_0)}_{\geq 0}$$

$\Rightarrow g_0 = 0$

By Prop, it remains to show that $\exists$ some divisor $A \in \text{Div}(F_0)$ st $\deg A = 1$

We know there exists $P_0 \in \mathbb{P}_{K(x)}$ st $\deg P_0 = 1$ (prop. 1.2.3) (i.e residue field $\mathcal{O}_{P_0}/P_0 = K$)

If $\tilde{P_0} \in \mathbb{P}_{F_0}$ st $P_0 \mid \tilde{P_0}$ then $1 \leq \deg \tilde{P_0} \leq \deg P_0$ $[F_{P_0}:K] = [F_{P_0}:F_{\tilde{P_0}}][F_{\tilde{P_0}}:K]$, $\deg P_0$, $\deg \tilde{P_0}$

$\Rightarrow \deg \tilde{P_0} = 1$

2. $K(x)/F_0$ not separable.

$$\underbrace{F_0 \subseteq F_1}_{\text{separable}} \underbrace{\subseteq K(x)}_{\text{purely insep.}}$$

$K \subset \underbrace{F_0 \subset}_{\text{sep}} F_1$ if we prove F_1/F_0 rational $\overset{A}{\Rightarrow}$ F_0 is rational.

$[K(x):F_1] = q = p^r$ with $p = \text{char}\, K > 0$ (otherwise separable) and $\forall z \in K(x)$ $z^q \in F_1$

$\Rightarrow K(x^q) \subseteq F_1 \underbrace{\subseteq}_{q} K(x)$

But $[K(x):K(x^q)] = \deg(x^q)_\infty = q$ (thm 1.4.11)

$\Rightarrow [F_1 : K(x^q)] = 1$

$\Rightarrow F_1 = K(x^q)$. $\square$

§3.6 Constant field extensions. (ie $F' = FK'$)

F/K, K perfect, $F \subset \Phi$ fixed alg. closed field

The goal here is to apply previous results + HG to determine the genus of a function field.

Let $K \subseteq K'$ an alg. extension ($K' \subseteq \Phi$)

$F' := FK'$ which is a function field over K'

Then K' is the full c^{st} field of F'. (Prop 3.6.1)

Goal: Apply previous results in this particular setting.

Theorem 3.6.3

Let $F' = FK'$ alg. const field extension of F/K.

(a) F'/F is unramified $\Rightarrow \mathrm{Diff}(F'/F) = 0 \Rightarrow 2g'-2 = \frac{[F':F]}{[K':K]}(2g-2) = 2g-2$, $\frac{[F':F]}{[K':K]} = 1$

(b) F'/K' has same genus as F/K.

(c) $A \in \mathrm{Div}(F) \leadsto \deg \mathrm{Con}_{F'/F}(A) = \deg(A)$.

(c) If W is a canonical divisor of F/K, i.e. $W = (\omega)$ for some W.D on F/K, then $\mathrm{Con}_{F'/F}(\omega)$ is a canonical divisor of F'/K'.

Corollary 3.6.5

$P \in \mathbb{P}_F$ of degree r, and let $\bar{F} = F\bar{K}$, where $\bar{K}$ the alg. closure of K.

Then

$$\mathrm{Con}_{\bar{F}/F}(P) = \sum_{P'|P} e(P'|P) \cdot P'$$

$$= \sum_{P'|P} P' \qquad \text{(by (a))}$$

(c) $\Rightarrow \deg \mathrm{Con}_{\bar{F}/F}(P) = \deg P = r$

$$= \bar{P}_1 + \dots + \bar{P}_r \quad \text{with } \bar{P}_r | P, \ \neq \text{ places}$$

□