

Firminaments

① Prerequisite on toric varieties

N a lattice

?

$\mathbb{Z}^n$

$$M := \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \simeq \mathbb{Z}^n$$

$$N_{\mathbb{k}} := N \otimes_{\mathbb{Z}} \mathbb{k}$$

$\uparrow$
a field

$$M_{\mathbb{k}} := M \otimes_{\mathbb{Z}} \mathbb{k}$$

$\mathbb{k}$ -vector spaces

$$\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$$

$$M_{\mathbb{k}} \times N_{\mathbb{k}} \rightarrow \mathbb{k}$$

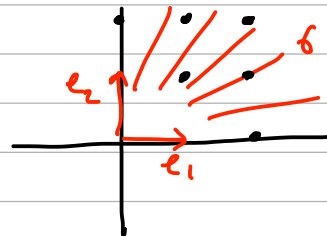
$\delta \subset N_{\mathbb{k}}$ is a (scrp) cone if $\exists v_1, \dots, v_s \in N$ st

$$\delta = \mathbb{R}_{\geq 0} v_1 + \dots + \mathbb{R}_{\geq 0} v_s = \operatorname{cone}(v_1, \dots, v_s)$$

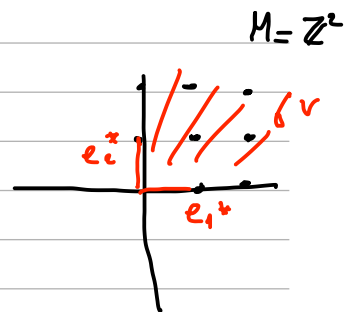
$$\text{dual of } \delta : \delta^{\vee} := \{m \in M_{\mathbb{k}} \mid \langle m, v \rangle \geq 0 \quad \forall v \in \delta\}$$

$$\text{Ex: } N = \mathbb{Z}^2$$

$$\delta = \operatorname{cone}(e_1, e_2)$$



$N = \mathbb{Z}^2$



$M \cap \delta^{\vee}$ is fg monoid

$\hookrightarrow \mathbb{k}[M \cap \delta^{\vee}]$ is a $\mathbb{k}$ -algebra of finite type.

$X_{\delta} := \operatorname{Spec} \mathbb{k}[M \cap \delta^{\vee}]$ the toric affine variety associated to δ in N .

Ex:

$$M \cap \delta^{\vee} = \mathbb{N}^2$$

$$X_{\delta} = \operatorname{Spec} \mathbb{k}[\mathbb{N}^2] \simeq \mathbb{A}_{x,y}^2 \quad (\operatorname{Spec} \mathbb{k}[x,y])$$

$\{0\}$ is a face (a cone itself) in δ

$$X_{\{0\}} = \text{Spec } k[M \cap \{0\}^\vee] = \text{Spec } k[M] = \text{Spec } k[\mathbb{Z}^n] \\ \cong \mathbb{G}_m^n \hookrightarrow X_\delta$$

ii
T the torus

in the previous example $X = \text{Spec } k[\mathbb{N}^2]$

$$T = \text{Spec } k[\mathbb{Z}^2] = \text{Spec } k[x, x^{-1}, y, y^{-1}] \\ \cong \mathbb{G}_m^2$$

- A fan Σ in N is a finite collection of cones in N st
 - each face of a cone is a cone
 - two cones meet in a face of both

if $\tau \leq \delta$ (a face), $X_\tau \hookrightarrow X_\delta$

Given a fan Σ , $U_{\delta_1}, U_{\delta_2}$

$\swarrow \quad \searrow$
 $U_{\delta_1 \cap \delta_2}$

glue

X_Σ a toric variety associated to Σ in N .

- T_N acts on X_Σ (extending its action on itself)

$$\{\text{cones of } \Sigma\} \xleftrightarrow{1-1} \{\text{orbits of the action}\}$$

$$\{\text{rays} = 1\text{-dim cones in } \Sigma\} \xleftrightarrow{1-1} \{\text{codim 1 orbits} : \mathcal{O}_\rho\}$$

the toric boundary of T_N is $\partial X_\Sigma = \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_\rho =: \bigoplus_{\rho \in \Sigma(1)} \overline{\mathcal{O}_\rho}$

- N_1, N_2 two lattices with fans Σ_1, Σ_2 resp.
- a map $f: N_1 \rightarrow N_2$ is compatible with the fans if
 - $\forall \delta_1 \in \Sigma_1, \exists \delta_2 \in \Sigma_2$ st $f(\delta_1) \subset \delta_2$.

$\Sigma_1 \hookrightarrow$

- $\varphi: X_1 \rightarrow X_2$ is a morphism of toric varieties iff it induces a linear map $f: N_1 \rightarrow N_2$ compatible with Σ_1 and Σ_2

Definition (toroidal embedding):

(1) A toroidal embedding $U \subset X$ is the data of a variety X and a dense open set U with complement a Weil divisor $D := X \setminus U$, such that locally in the étale topology, near every closed point $x \in X$, $U \subset X$ admits an isomorphism with a neighborhood of a point in $T \subset V$, with T a torus and V a toric variety.

$\forall x \in X$ closed, there exists a toric variety X_0 , a point $s \in X_0$ and an isomorphism of local algebras

$$\hat{\mathcal{O}}_{X,x} \simeq \hat{\mathcal{O}}_{X_0,s}$$

st the ideal in $\hat{\mathcal{O}}_{X,x}$ generated by the ideal of $X \setminus U$ corresponds under this isomorphism to the ideal of $\hat{\mathcal{O}}_{X_0,s}$ generated by the ideal of $X_0 \setminus T$.

the pair (X_0, s) , together with the isomorphism is called a local model at $x \in X$. strict toroidal embedding = the inv. comp of D are normal

($\Rightarrow$) $\forall x \in X$, $\exists$ Zariski neighborhood $\phi_u: V_u \hookrightarrow X$ containing U_x and an étale morphism $\psi_u: V_u \rightarrow X_0$ st $\psi_u^{-1}(T) = U_x$

$$U \subset V_u \xrightarrow{\phi_u} X \supset U$$

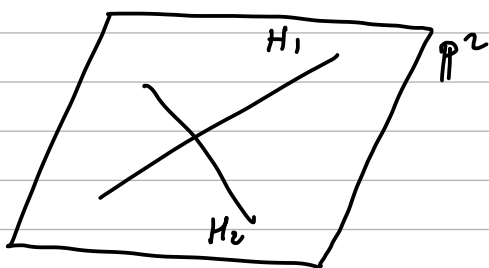
$$\psi_u \downarrow \text{étale}$$

$$T \subset X_0 \text{ (affine toric)}$$

Examples:

- Any toric variety is a toroidal embedding.

- $X = \mathbb{P}^2$, $U = X \setminus H_1 \cup H_2$ $H_i = \{u_i = 0\}$



Remarks: (X, D) a toroidal embedding

1) if $D = \bigcup_{i \in I} E_i$
you can define a stratification of X

• the components of $\bigcup_{i \in J} E_i \setminus \bigcup_{i \notin J} E_i$ (JCI) (orbits if X is toric variety)

if Y is a strata, $\text{Star } Y := \bigcup_{\substack{Z \text{ strata} \\ Y \subset \bar{Z}}} Z$ is an open set.

(Ex, if $\dim X = 2$, E_i are curves $\leadsto$ the singularities are ordinary double point (nodes) i.e. $D = \bigcup E_i$ is a(s) normal crossing divisor and the stratification is given by

- $\bigcup$
- the components of the sets E_i - nodes of $\bigcup E_i$
- the nodes of $\bigcup E_i$.

2) Let (X_δ, s) be a local model at $x \in X$.

$$I = \{ \text{ray of } \delta \}$$

($O(\delta)$ orbits of codim = 1 in X_δ)

$$I_u = \{ i \in I \text{ st } x \in E_i \}$$

$$\text{then } \text{Spec } \hat{\mathcal{O}}_{X,u} \simeq \text{Spec } \hat{\mathcal{O}}_{X_\delta, s}$$

induces an isomorphism of I_u and I' under which E_i corresponds to $\hat{\mathcal{O}}(p)$.

$\rightarrow$ is a Weil divisor on X_δ

$$\begin{array}{c} E_i \hookrightarrow X \\ \uparrow \\ \text{Spec } \hat{\mathcal{O}}_{X,u} \\ \uparrow \\ \text{Spec } \hat{\mathcal{O}}_{X,u} \end{array}$$

$$\begin{array}{c} \hat{\mathcal{O}}(p) \hookrightarrow X_\delta \\ \uparrow \\ \text{Spec } \hat{\mathcal{O}}_{X_\delta, s} \\ \uparrow \\ \text{Spec } \hat{\mathcal{O}}_{X_\delta, s} \end{array}$$

3) Let (X_δ, s) be a local model at $x \in X$.
 Then δ does not depend of x but only of the strata containing x .

Indeed, let Y the strata $\ni x$.

$M^Y =$ gp of Cartier divisors on $\text{Star}(Y)$ with support on $\text{Star}(Y) - U$

$$M^Y_k = M^Y \otimes k$$

$$N^Y = \text{Hom}(M^Y, \mathbb{Z}), \quad N^Y_k = N^Y \otimes k = \text{Hom}(M^Y_k, k)$$

$M^Y_+ \subset M^Y$, sub monoid of effective Cartier divisors

$$\delta^Y = \{u \in N^Y_k \mid \langle m, u \rangle \geq 0 \quad \forall m \in M^Y_+\} \subset N^Y_k$$

(dual of M^Y_+)

if $\text{Star } Y - U = \bigcup_{i \in J} E_i$, M^Y is a subgroup of the free abelian group of Weil divisors $\sum_{i \in J} n_i E_i$ on $\text{star } Y$.

Thus M^Y and N^Y are finitely generated free abelian gps.
 We have the following:

if M is the lattice associated to the abelian toric variety X_δ ,

$$M \longrightarrow M^Y$$

$$m \mapsto \sum n_i E_i \quad \text{if} \quad \text{div}(X^m) = \sum n_i \bar{E}_i$$

$$\left(\text{with } E_i \xrightarrow{1-1} \bar{E}_i \right. \\ \left. \hookrightarrow 1\text{-codim orbit} \right)$$

$$\begin{aligned} m \in M &= \text{Hom}(T_N, G_m) \\ &= \text{Hom}(k[T, T^{-1}], k[X_1, X_1^{-1}, \dots]) \\ &\leadsto \in \mathcal{O}_{T_N} (T_N)^{\times 2} := X^m \end{aligned}$$

- well-defined: $\sum n_i \bar{E}_i$ Cartier $\Leftrightarrow \sum n_i E_i$ Cartier (II, lemma 1)
- surjective: if $\sum n_i E_i$ is Cartier $\leadsto \sum n_i \bar{E}_i$ is Cartier

$$\begin{aligned} M &\longrightarrow \mathcal{W}\text{div}(X_\delta) \longrightarrow \mathcal{C}\ell(X_\delta) \longrightarrow 0 \\ m &\mapsto \text{div}(X^m) \end{aligned}$$

But $\text{Pic}(X_\delta) = 0 \Rightarrow \exists m \in M$ st $\sum n_i \bar{\theta}_i = \text{div}(X^m)$

- Kernel? $\{m \in M \mid \text{div}(X^m) = 0\}$

$$\sum \langle m, u_i \rangle \bar{\theta}_i$$

minimal generators of rays
(generate the cone δ)
 $\hookrightarrow m \in \delta^\perp$

$$\text{hence } M / \delta^\perp \cong M^\vee$$

$$N \cap \text{span}(\delta) \cong N^\vee$$

$$\delta \cong \delta^\vee$$

Definition (Cone complex)

To each toroidal embedding, we attach an integral polyhedral cone complex Σ_X , consisting of cones (the ones that appear with the local models), attached to each other along faces.

We identify the cones δ to $\delta^\vee$ and see Σ_X as the collection of $\delta^\vee$.

(Note that Σ_X do not lie linearly inside an ambient space of the form $N \otimes \mathbb{R}$)

Definition (Valuation rings and the cone complex):

Let (X, U, D) be a toroidal embedding (for each valuation on X , we are going to attach an element from a lattice)

Let R be a dvr and $\phi: \text{Spec } R \rightarrow X$

Q: Why is this condition satisfied if $R = R_v$? $\phi(\eta_R) \subset U$ and $\phi(s_R)$ lie over some stratum Y having étale chart X_δ (with lattice M).

(think about R being R_v , v a valuation on Y)

We want to associate an element of $N_\delta: M_\delta \rightarrow \mathbb{Z}$
 $\delta \cap N \quad \delta^\vee \cap M$

Each $m \in M_0 \subset M \rightarrow M^Y$, gives a Cartier div Δ_m on $\text{star } Y$.

$$\begin{array}{ccc} \text{Spec } K & \xrightarrow{\phi} & X \\ \uparrow & & \uparrow \\ \mathbb{A}^1_{\overline{K}} & \xrightarrow{\phi^* \Delta_m} & \Delta_m \end{array}$$

then $v(\phi^*(\Delta_m)) \in \mathbb{Z}$

This gives a map $M_0 \rightarrow \mathbb{Z}$, hence an element of N_0 .

If $R = K^*$, we denote this element n_v .

Definition (morphism of toroidal embeddings) :

Let $(U_X \subset X)$ and $(U_Z \subset Z)$ be toroidal embeddings. A dominant morphism $f: X \rightarrow Z$ is said to be toroidal if étale locally, near every closed point $x \in X$, there is a toric chart for X near x , and a toric chart for Z near $f(x)$ such that on these charts, f becomes a toric map of toric varieties.

i.e. $\forall x \in X$ closed, there exists local models (X_δ, s) at x , (Z_τ, t) at $f(x)$ and a toric morphism $g: X_\delta \rightarrow Z_\tau$, so that the following diagram commutes

$$\begin{array}{ccc} \wedge & & \wedge \\ \mathcal{O}_{X_\delta, x} & \xleftarrow{\simeq} & \mathcal{O}_{X_\delta, s} \\ \uparrow \hat{f}^* & & \uparrow \hat{g}^* \\ \mathcal{O}_{Z_\tau, f(x)} & \xleftarrow{\simeq} & \mathcal{O}_{Z_\tau, t} \end{array}$$

Lemma :

A toroidal morphism $f: X \rightarrow Z$ induces a morphism $f_\Sigma: \Sigma_X \rightarrow \Sigma_Z$, which for each cone $\delta' \in \Sigma_X$ is

the restriction of $g_\Sigma: N^{\delta'} \rightarrow N^{\tau}$ where $(X_\delta, s, N^{\delta'})$ and (Z_τ, t, N^{τ}) are local models at Σ $x \in Y \subset X$, $f(x) \in Y' \subset Z$, and g_Σ linear map determined by the toric morphism $g: X_\delta \rightarrow Z_\tau$.

In particular, $f_{\Sigma}(N^{\circ}) \subset N^{\circ}$ for each $f(\delta) \subset \Sigma$.

Proof: what is not clear is if we take another $u' \in X$ st $f(u') \notin \Sigma$...

the dual $f_{\Sigma}^{\vee} : M^{\vee} \rightarrow M^{\vee}$ away from U_{Σ}

Q: Same as pull back C-D on star Σ ?
is defined by pulling back a Cartier divisor and restrict it to star γ : it gives a Cartier divisor defined away from U_X . But the pull-back doesn't depend of the strata, so f_{Σ} is well defined.

□

Theorem (Toroidalizing a morphism): Abramovich-Karu

Let $f: X \rightarrow Y$ be a dominant morphism of varieties. Then there exists modifications $X' \rightarrow X$ and $Y' \rightarrow Y$ and toroidal structures $U_{X'} \subset X'$, $U_{Y'} \subset Y'$ st the resulting rational map $f': X' \rightarrow Y'$ is a toroidal map.

$$\begin{array}{ccccc} U_{X'} & \hookrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow f' & & \downarrow f \\ U_{Y'} & \longrightarrow & Y' & \longrightarrow & Y \end{array}$$

furthermore, f' can be chosen flat.

Definitions:

Toroidal firmament:

Let $(U \subset X)$ be a toroidal embedding with complex Σ_X . A toroidal firmament on X is a finite collection $\Gamma = \{\Gamma_{\delta}^i \subset N_{\delta}^{\circ}\}$ with $\delta \in \Sigma_X$ where.

- each $\Gamma_{\delta}^i \subset N_{\delta}^{\circ}$ is fg submonoid
- each Γ_{δ}^i generates δ as a cone.
- the collection is closed under restrictions to faces, i.e. for each Γ_{δ}^i , $\tau \triangleleft \delta$, $\exists j$ st $\Gamma_{\delta}^i \cap \tau = \Gamma_{\tau}^j$.
- it is irredundant ($\Gamma_{\delta}^i \not\subset \Gamma_{\tau}^j$ for i, j).

Morphism of toroidal firmaments:

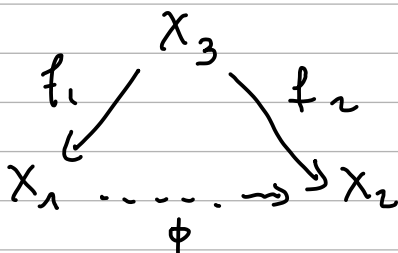
a morphism from a toroidal firmament Γ_X on $U_X \subset X$ to Γ_Y on $U_Y \subset Y$ is a morphism $f: X \rightarrow Y$ with $f(U_X) \subset U_Y$ st $\forall \delta \in \Sigma_X, \forall i$, if $f(\delta) \subset \tau$, we have $f(\Gamma_\delta^i) \subset \Gamma_\tau^j$ for some j .

Induced toroidal firmament:

We say that a toroidal firmament Γ_X is induced by $f: X \rightarrow Y$ from Γ_Y if $\forall \delta \in \Sigma_X$ and $\tau \in \Sigma_Y$ st $f(\delta) \subset \tau$ we have $\Gamma_\delta^i = f^{-1}(\Gamma_\tau^j) \cap N_\delta$ for some j .

↪

- Given a proper birational equivalence $\phi: X_1 \dashrightarrow X_2$ then two toroidal firmaments Γ_{X_1} and Γ_{X_2} are said to be equivalent if there is a toroidal embedding $U_3 \subset X_3$ and a commutative diagram:



where f_i are modifications sending U_3 to U_i and such that the two toroidal firmaments on X_3 induced by f_1 and f_2 from Γ_{X_i} are identical.

firmament on any var
morphism of firmament

- A firmament on an arbitrary X is the same as an equivalence class represented by a modification $X' \rightarrow X$ with $U' \subset X'$ a toroidal embedding and a firmament Γ on X' . A morphism of firmaments is a morphism of varieties which becomes a morphism of toroidal firmaments on some toroidal model.

- The trivial firmament is defined by $\Gamma_\delta^i = N_\delta \quad \forall \delta \in \Sigma$.

Definition (Base firmament)

- 1) $f: X \rightarrow Y$ a flat toroidal morphism of toroidal embeddings.
The base firmament Γ_f associated to f is defined by the images

$$f_\Sigma(\Gamma_\delta) := \Gamma_\delta^\tau \quad \text{for each cone } \delta \text{ over } \tau \quad \begin{matrix} f(\delta) \subset \tau \\ \delta \in \Sigma_X \quad \tau \in \Sigma_Y \end{matrix}$$

- 2) Let $f: X \rightarrow Y$ a dominant morphism of varieties.

The base firmament of f is represented by any $\Gamma_{f'}$ where $f': X' \rightarrow Y'$ is a flat toroidal model of f .
(This does not depend of the model)

- 3) If X reducible, $X = \cup X_i$ and $f: X_i \rightarrow Y$ dominant for each i , we define the base firmament by the (maximal elements) of the union of all firmaments associated to $X_i \rightarrow Y$.

Is not clear if same toroidal structure on Y .

Valuative firmament / equivalence of definitions:

Let X be a variety, X' a toroidal model and Γ a firmament on X' . Let v a valuation on X , we have $v_\sigma \in N\sigma$.
(for some σ).

We define $\Gamma_v = \{ \sigma \in N \mid v_\sigma \in \Gamma_\sigma \text{ for some } i \}$.

This gives a firmament in the valuative definition.

Conversely, a firmament in the valuative definition has finitely many étale charts $Y_i \rightarrow Y$ where the firmament comes from $X_i \rightarrow Y_i$. One can toroidalize each $X_i \rightarrow Y_i$ simultaneously over some toroidal structure $U \subset Y$.

if $U \subset Y$ toroidal embedding, then $f_i^{-1}(U) \hookrightarrow Y_i$ is also one.

$$\begin{array}{ccc} f_i^{-1}(U) \subset Y_i & \xrightarrow{\text{étale}} & Y \supset U \\ \uparrow \eta & \searrow f_i & \uparrow \eta \\ Z_i & \xrightarrow{f_i} & Z_i \end{array}$$

$$\begin{array}{ccc} f_i^{-1}(Z_i) & \xrightarrow{\text{étale}} & Z_i \xrightarrow{\text{étale}} X_0 \supset T \\ \uparrow \eta & & \uparrow \eta \\ Z_i & \xrightarrow{\eta} & Z_i \end{array}$$

$f_i^{-1} \circ \eta_i^{-1}(T) = f_i^{-1}(U)$

So "simultaneously" means we consider the toroidal embedding $f_i^{-1}(U) \hookrightarrow Y_i$ $\forall i$ and toroidalyze the $X_i \xrightarrow{f_i} Y_i$

The two procedures are inverse to each other. In particular, this shows that any firmament supports a unique constellation.

$$(\text{firmament} \xrightarrow{1-1} \text{valuative firmament} \rightsquigarrow \Delta = (1 - \frac{1}{m_v}) v)$$

$m_v = \min \pi_v$

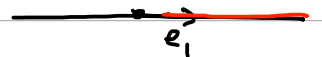
Examples:

(1) $f: A^2 \rightarrow A^1$ is a toric map of toric varieties

$$\begin{array}{ccc} k[x, y] & \hookrightarrow & k[t] \\ u^2 & \hookrightarrow & t \end{array}$$

Let's compute Γ_f .

A^1 is an affine toric variety



$$\delta = \text{cone}(e_1) \quad N_\delta = \mathbb{Z} \quad M_\delta = N^\vee \simeq \mathbb{Z}$$

$$N_\delta = \delta \cap N = \mathbb{N}$$

$$M_\delta = \delta^\vee \cap N = \mathbb{N}$$

$$A^1 = \text{Spec } k[M_\delta] = \text{Spec } k[\mathbb{N}] \simeq \text{Spec } k[t] \quad (1 \mapsto t)$$

A^2 is an affine toric variety

$$\delta' = \text{cone}(e_1, e_2) \quad N_{\delta'} = \mathbb{Z}^2, \quad M_{\delta'} = \mathbb{Z}^2$$

$$N_{\delta'} = \mathbb{N}^2$$

$$M_{\delta'} = \mathbb{N}^2$$

$$\begin{array}{l} (0, 1) \mapsto y \\ (1, 0) \mapsto x \end{array}$$

$$A^2 = \text{Spec } k[M_{\delta'}] = \text{Spec } k[\mathbb{N}^2] \simeq \text{Spec } k[x, y]$$

$$f: A^2 \rightarrow A^1 \rightsquigarrow f_\Sigma: \begin{array}{ccc} \mathbb{N}^2 & \longrightarrow & \mathbb{N} \\ N_\delta & & N_1 \\ (a, b) & \longmapsto & 2a \end{array}$$

$$\text{So } \Gamma_f := \{f_\Sigma(\mathbb{N}^2) = 2\mathbb{N}\}$$

$$e_1^\perp = \{u \in e_1^\perp \text{ st } \langle u, e_1 \rangle = 0\} = 0$$

" $M = M^\perp$ "

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2

$$\text{Spec } k[t]_{(t-a)} \cong k[t]_t \simeq \frac{k[t]_{(t-a)}}{(t)} \rightarrow D = \{t=0\} = \text{Spec } k[t]_t = \text{Spec } k$$

if $a \neq 0$, $n_\sigma = 0 \in \mathbb{N} \rightsquigarrow \Gamma_\sigma = \mathbb{N} \rightsquigarrow m_\sigma = 1 \rightsquigarrow \delta_\sigma = 0$
 if $a = 0$, $n_\sigma = 1 \in \mathbb{N} \rightsquigarrow \Gamma_\sigma = \{k \in \mathbb{N} \mid k \in 2\mathbb{N}\}$
 $m_\sigma = \min \Gamma_\sigma \setminus \{0\} = 2 \rightsquigarrow \delta_\sigma = 1/2$

The diagram illustrates the relationship between two sets, A^2 and A^1 . On the left, a coordinate system is shown with axes labeled $a=0$ and $a=1$. The vertical axis is labeled $\mathbb{N}^2$ and the horizontal axis is labeled $2a$ and $\mathbb{N}$. A grid of points is plotted, representing elements of A^2 . A horizontal line is drawn through the grid, labeled $2a$ and $\mathbb{N}$. On the right, a vertical arrow points from A^2 down to A^1 , with a f next to it, indicating a mapping function.

$$f_2: \mathbb{N}^2 \rightarrow \mathbb{N}$$

$$(a, b) \mapsto 2a + b$$

here $\Gamma_v = \{k \in \mathbb{N} \mid k_{uv} \in \mathbb{N}\} \rightarrow$ empty condition!
 $= \mathbb{N}$

$$\Rightarrow \Delta p = 0$$

$$4) f: A^2 \rightarrow A^1 \\ (u^2, y^3) \mapsto t$$

$$f_{\Sigma}: \mathbb{N}^2 \rightarrow \mathbb{N} \\ (a, b) \mapsto 2a + 3b$$

$$\Gamma = \{ f(\mathbb{N}^2) = 2\mathbb{N} + 3\mathbb{N} \}$$

$$n_v = \begin{cases} 1 & \text{if } a=0 & (t) \\ 0 & \text{if } a \neq 0 & (t-a) \end{cases}$$

$$\Gamma_v = \{ h \in \mathbb{N} \mid h_{n_v} \in 2\mathbb{N} + 3\mathbb{N} \} \\ a=0 \\ = \{ h \in \mathbb{N} \mid h \in 2\mathbb{N} + 3\mathbb{N} \}$$

$$m_v = \min \Gamma_v = "h = 2" \rightsquigarrow \delta_v = 1/2$$

$$\Delta \Gamma = 1/2 \quad (t=0) \quad (\text{Same constellation as ex 1})$$

$$6) f: A^2 \rightarrow A^2 \\ (u^2, y) \mapsto (s, t)$$

$$f_{\Sigma}: \mathbb{N}^2 \rightarrow \mathbb{N}^2$$

$$(a, b) \mapsto (2a, b)$$

$$\Gamma_f = \{ f(\mathbb{N}^2) = 2\mathbb{N} \times \mathbb{N} \}$$

$$\delta^{\perp} = \text{cone}(\mathbf{e}_1, \mathbf{e}_2)^{\perp} = \\ \{ m \in M_{\mathbb{R}} \mid \langle m, \mathbf{e}_i \rangle > 0 \}$$

Let v be divisorial valuation on A^2 .

$$\text{Spec } R_v' \xrightarrow{\phi} A^2 = \text{Spec } k[s, t]$$

$$(\text{Spec } R_v' = Y_v')$$

$$\text{Spec } R_v' / s$$

$$D = \{ s=0 \} \\ D' = \{ t=0 \}$$

$$\downarrow Y_v' \\ \downarrow A^2)$$

$$m \in M \simeq \mathbb{Z}^2$$

$$n_v := (v(s), v(t)) \in \mathbb{N}^2$$

$$\uparrow (1, 0) \sim s \\ \downarrow (0, 1) \sim t$$

$$\Gamma_v = \{ k \in \mathbb{N} \mid k_{uv} \in 2\mathbb{N} \times \mathbb{N} \}$$

$$(k_v(s), k_v(t))$$

$$\text{no if } v(s) = 2a, m_v = 1 \leadsto \delta_v = 0 \leadsto \Delta \pi = \dots$$

$$\text{if not } m_v = 2, \delta_v = 1/2$$

Arithmetics:

Definition: Let $k, O_k, S \subset \text{Spec } O_k$

An S -integral model of a toroidal firmament Γ on Y is an integral toroidal model Y' of Y .

- Consider a toroidal firmament Γ on Y/k and $y \in Y(k)$ such that Γ is trivial in a neighborhood of y .

$$\text{Spec } k \xrightarrow{y} Y$$

$$* \mapsto \text{strata} \leadsto \sigma$$

$$\text{we want that } \Gamma^i \in N_\sigma$$

$$\parallel$$

$$N_\sigma$$

Let Y be an S -integral model.

Then y is a firm integral point of Y with respect to Γ if the section $\text{Spec } O_{k,S} \rightarrow Y$ (exists?) is a morphism of firmaments, with $\text{Spec } O_{k,S}$ given the trivial one.

toroidal embedding?

Explicitly, at each prime $p \in \text{Spec } O_{k,S}$, where y reduces to a stratum with cone σ , consider the associated point $n_{y,p} \in N_\sigma$. Then y is firmly S -integral if $\forall p, n_{y,p} \in \Gamma^i_\sigma$ for some i .

$$b_\tau : \mathbb{N} \rightarrow N_\sigma$$

$$\text{map of firmaments} \Leftrightarrow b_\tau(\mathbb{N}) \subset \Gamma^i_\sigma$$

$$\Leftrightarrow b_\tau(1) \in \Gamma^i_\sigma$$

$$\parallel$$

$$n_{y,p}$$

$$\text{Spec } O_{k,S,p} \xrightarrow{\text{valuation}} Y$$

$$\text{Spec } \mathcal{O}_{d,S,p} \rightarrow Y$$

$$sp \mapsto \text{strata } H \text{ with cone } \delta \xrightarrow{\sim} N\delta$$

$$\delta^H \quad M\delta \rightarrow M^H$$

n_p is the "collection" of degree
of tangencies of Cartier divisors on $\text{Star } H$
and y .

Theorem:

$f: X \rightarrow Y$ proper dominant morphism of var/k.

There exists a toroidal birational model $X' \rightarrow Y$ and
an integral model Y' st the image of a rational point
on X' is a firm S -integ point on Y' with respect to Γ_f .

- Maybe after enlarging S , a point is firm S -integ on Y'
with respect to Γ_f iff locally in the étale top on Y' ,
it lifts to a rational point on X .

Conjecture

Let (Y/S) be a smooth proj Campana Constellation
supported by a firmament Γ .

Then points on Y Γ -integral with respect to Γ are potentially
dense iff (Y/S) is special.

$\hookrightarrow$ implies conj *