# Firmaments

## ① Prerequise on toric varieties

$N$ a lattice
$\wr\wr$
$\mathbb{Z}^n$
$M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \simeq \mathbb{Z}^n$

$N_k := N \underset{\mathbb{Z}}{\otimes} k \qquad M_k := M \underset{\mathbb{Z}}{\otimes} k \qquad$ $k$-vecta spaces

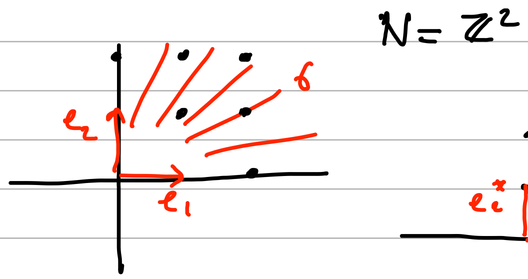
↗ a field

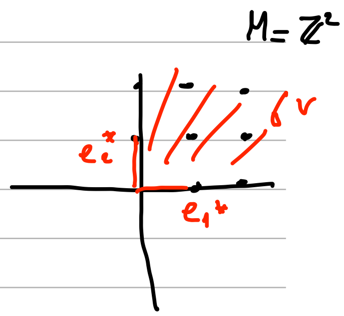
$$\langle \cdot , \cdot \rangle : M \times N \longrightarrow \mathbb{Z}$$
$$M_k \times N_k \to \mathbb{Z}$$

$\sigma \subset N_k$ is a (scrp) cone if $\exists\, v_1, \dots, v_s \in N$ st
$$\sigma = \mathbb{R}_{\geq 0} v_1 + \dots + \mathbb{R}_{\geq 0} v_s = \mathrm{cone}(v_1, \dots, v_s)$$

dual of $\sigma$ : $\sigma^\vee := \{ m \in M_{\mathbb{R}} \mid \langle m, v \rangle \geq 0 \ \forall v \in \sigma \}$

Ex: $N = \mathbb{Z}^2$
$\sigma = \mathrm{cone}(e_1, e_2)$




$M \cap \sigma^\vee$ is fg monoid

↪ $k[M \cap \sigma^\vee]$ is a $k$-algebra of finite type.

$X_\sigma := \mathrm{Spec}\, k[M \cap \sigma^\vee]$ the toric affine variety associated to $\sigma$ in $N$.

Ex:

$M \cap \sigma^\vee = \mathbb{N}^2$
$X_\sigma = \mathrm{Spec}\, k[\mathbb{N}^2] \simeq \mathbb{A}^2_{x,y}$ $(\mathrm{Spec}\, k[x,y])$

$\{0\}$ is a face (a cone itself) in $\sigma$

$$X_{\{0\}} = \mathrm{Spec}\, k[M \cap \{0\}^\vee] = \mathrm{Spec}\, k[M] = \mathrm{Spec}\, k[\mathbb{Z}^n] \cong \mathbb{G}_m^n \hookrightarrow X_\sigma$$

$\parallel$

$T$ the torus

in the previous example $X = \mathrm{Spec}\, k[\mathbb{N}^2]$

$$T = \mathrm{Spec}\, k[\mathbb{Z}^2] = \mathrm{Spec}\, k[x, x^{-1}, y, y^{-1}] \cong \mathbb{G}_m^2$$

- A fan $\Sigma$ in $N$ is a finite collection of cones in $N$ st
  - each face of a cone is a cone
  - two cones meet in a face of both

if $\tau < \sigma$ (a face), $X_\tau \hookrightarrow X_\sigma$

Given a fan $\Sigma$, $U_{\sigma_1}$, $U_{\sigma_2}$

$$U_{\sigma_1} \hookleftarrow U_{\sigma_1 \cap \sigma_2} \hookrightarrow U_{\sigma_2}$$

glue

$X_\Sigma$ a toric variety associated to $\Sigma$ in $N$.

- $T_N$ acts on $X_\Sigma$ (extending its action on itself)

$$\{\text{cones of } \Sigma\} \overset{1-1}{\longleftrightarrow} \{\text{orbits of the action}\}$$

$$\{\text{rays} = 1\text{-dim cones in } \Sigma\} \overset{1-1}{\longleftrightarrow} \{\text{codim } 1 \text{ orbits} : O_\rho\}$$
$\rho$

- the toric boundary of $T_N$ is $\partial X_\Sigma = \bigoplus_{\rho \text{ a ray}} D_\rho =: \bigoplus_{\rho \text{ rays}} \overline{O}_\rho$

- $N_1$, $N_2$ two lattices with fans $\Sigma_1$, $\Sigma_2$ resp.

a map $f : N_1 \to N_2$ is compatible with the fans if $\forall \sigma_1 \in \Sigma_1$, $\exists\, \sigma_2 \in \Sigma_2$ st $f(\sigma_1) \subset \sigma_2$.

$\Sigma_1 \hookrightarrow$ $\Sigma_2$

- $\varphi : X_1 \to X_2$ is a morphism of toric varieties iff it induces a linear map $f : N_1 \to N_2$ compatible with $\Sigma_1$ and $\Sigma_2$

## Definition (toroidal embedding):

(1) A toroidal embedding $U \subset X$ is the data of a variety $X$ and a dense open set $U$ with complement a weil divisor $D := X \setminus U$, such that locally in the étale topology, near every closed point $x \in X$, $U \subset X$ admits an isomorphism with a neighborhood of a point in $T \subset V$, with $T$ a torus and $V$ a toric variety.

$\forall x \in X$ closed, there exists a toric variety $X_\sigma$, a point $s \in X_\sigma$ and an isomorphism of local algebras

$$\hat{O}_{X,x} \simeq \hat{O}_{X_\sigma,s}$$

st the ideal in $\hat{O}_{X,x}$ generated by the ideal of $X \setminus U$ corresponds under this isomorphism to the ideal of $\hat{O}_{X_\sigma,s}$ generated by the ideal of $X_\sigma \setminus T$.

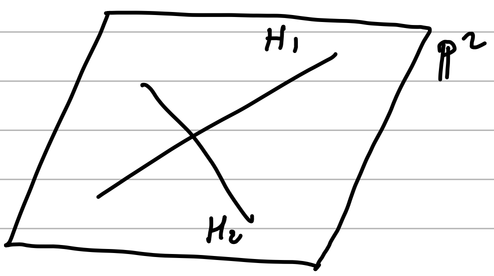
The pair $(X_\sigma, s)$, together with the isomorphism is called a local model at $x \in X$. strict toroidal embedding = the irred comp of $D$ are normal

$(\Leftrightarrow)$ $\forall x \in X$, $\exists$ Zariski neighborhood $\phi_x : V_x \hookrightarrow X$ containing $U_x$ and an étale morphism $\psi_x : V_x \to X_\sigma$ st $\psi_x^{-1}(T) = U_x$

$$\begin{tikzcd} U \subseteq V_x \arrow[r, hook, "\phi_x"] \arrow[d, "\psi_x"', "\text{étale}"] & X \supseteq U \\ T \subseteq X_\sigma \ (\text{affine toric}) & \end{tikzcd}$$

## Examples:

- Any toric variety is a toroidal embedding.
- $X = \mathbb{P}^2$, $U = X \setminus H_1 \cup H_2$ $\qquad H_i = \{x_i = 0\}$



**Remarks:** $(X, D)$ a toroidal embedding

1) if $D = \bigcup_{i\in I} E_i$
you can define a stratification of $X$

$\bigcup$ • the component of $\bigcap_{i\in J} E_i \setminus \bigcup_{i\notin J} E_i$ $\quad (J \subset I)$ $\quad$ (orbits if $X$ is toric variety)

if $Y$ is a strata, $\operatorname{Star} Y := \bigcup_{\substack{Z \text{ strata} \\ Y \subset \overline{Z}}} Z$ is an open set.

(Ex, if $\dim X = 2$, $E_i$ are curves $\rightsquigarrow$ the singularities are ordinary double point (nodes) i.e $D = \bigcup E_i$ is a(s) normal crossing divisor and the stratification is given by
$\bigcup$
- the component of the sets $E_i$ − nodes of $\bigcup E_i$
- the nodes of $\bigcup E_i$. )

2) Let $(X_\sigma, s)$ be a local model at $x \in X$.

$I' = \{ \rho \text{ ray of } \sigma \}$

$(O(\rho)$ orbits of codim $= 1$ in $X_\sigma)$

$I_x = \{ i \in I \text{ st } x \in E_i \}$

then $\operatorname{Spec} \hat{O}_{X,x} \simeq \operatorname{Spec} \hat{O}_{X_\sigma, s}$

induces an isomorphism of $I_x$ and $I'$ under which $E_i$ corresponds to $\overline{O(\rho)}$. $\rightsquigarrow$ is a Weil divisor on $X_\sigma$

$$\begin{tikzcd} E_i \arrow[r, hook] & X \\ & \operatorname{Spec} O_{X,x} \arrow[u] \\ & \operatorname{Spec} \hat{O}_{X,x} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \overline{O(\rho)} \arrow[r, hook] & X_\sigma \\ & \operatorname{Spec} O_{X_\sigma,s} \arrow[u] \\ & \operatorname{Spec} \hat{O}_{X_\sigma,s} \arrow[u] \end{tikzcd}$$

3) Let $(X_\sigma, s)$ be a local model at $x \in X$.
Then $\sigma$ does not depend of $x$ but only of the strata containg $x$.

Indeed, let $Y$ the strata $\ni x$.

$M^Y =$ gp of Cartier divisors on $\mathrm{Star}(Y)$ with support on $\mathrm{Star}\, Y - U$

$M^Y_k = M^Y \otimes k$

$N^Y = \mathrm{Hom}(M^Y, \mathbb{Z})$, $N^Y_k = N^Y \otimes k = \mathrm{Hom}(M^Y_k, k)$

$M^Y_+ \subset M^Y$, sub monoide of effective Cartier divisors

$\sigma^Y = \{ x \in N^Y_k \mid \langle m, x \rangle \geq 0 \ \forall m \in M^Y_+ \} \subset N^Y_k$

(dual of $M^Y_+$)

if $\mathrm{Star}\, Y - U = \bigcup_{i \in J} E_i$, $M^Y$ is a subgp of the free abelian group of Weil divisors $\sum_{i \in J} n_i E_i$ on $\mathrm{star}\, Y$.

Thus $M^Y$ and $N^Y$ are finitely generated free abelian gps.
We have the following:

if $M$ is the lattice associated to the affine toric variety $X_\sigma$,

$$M \twoheadrightarrow M^Y$$

$$m \mapsto \sum n_i E_i \quad \text{if} \quad \mathrm{div}(\chi^m) = \sum n_i \overline{\theta_i}$$

(with $E_i \overset{1-1}{\longleftrightarrow} \theta_i$ ) $\hookrightarrow$ 1-codim orbit

$m \in M = \mathrm{Hom}(T_N, \mathbb{G}_m)$
$= \mathrm{Hom}(k[T, T^{-1}], k[X, X^{-1} \ldots])$
$\rightsquigarrow \in \mathcal{O}_{T_N}(T_N)^* := \chi^m$

- well-defined: $\sum n_i \overline{\theta_i}$ Cartier $\Leftrightarrow$ $\sum n_i E_i$ Cartier (II, lemma 1)
- Surjective: if $\sum n_i E_i$ is Cartier $\rightsquigarrow \sum n_i \overline{\theta_i}$ is Cartier

$$M \to \mathrm{Wdiv}(X_\sigma) \to Cl(X_\sigma) \to 0$$
$$m \mapsto \mathrm{div}(\chi^m)$$

But $\mathrm{Pic}(X_\sigma) = 0 \Rightarrow \exists\, m \in M$ st $\sum n_i \bar{D}_i = \mathrm{div}(x^m)$

- Kernel? $\{ m \in M \mid \mathrm{div}(x^m) = 0 \}$

$$\mathrm{div}(x^m) = \sum \langle m, u_i \rangle \bar{D}_i$$

$u_i$: minimal generators of rays (generate the cone $\sigma$)

$\Rightarrow m \in \sigma^\perp$

hence $M/_{\sigma^\perp} \cong M^Y$

$$N \cap \mathrm{span}\langle \sigma \rangle \cong N^Y$$
$$\cup\!\!\!\text{|}$$
$$\sigma \simeq \sigma^Y$$

## Definition (Cone complex)

To each toroidal embedding, we attach an integral polyhedral cone complex $\Sigma_X$, consisting of cones (the ones that appear with the local models)., attached to each other along faces.
We identify the cones $\sigma$ to $\sigma^Y$ and see $\Sigma_X$ as the collection of $\sigma^Y$.
(Note that $\Sigma_X$ do not lie linearly inside an ambient space of the form $N_\mathbb{R}$)

## Definition (Valuation rings and the cone complex):

Let $(X, U, D)$ be a toroidal embedding (for each valuation on $X$, we are going to attach an element from a lattice)
Let $R$ be a dvr and $\phi : \mathrm{Spec}\, R \to X$

Q: Why is this condition satisfied if $R = R_v$?

$\phi(\eta_R) \subset U$ and $\phi(s_R)$ lie over some stratum $Y$ having étale chart $X_\sigma$ (with lattice $M$).

(Think about $R$ being $R_v$, $v$ a valuation on $Y$)

We want to associate an element of $N_\sigma : M_\sigma \to \mathbb{Z}$

$N_\sigma = \sigma \cap N$, $\quad M_\sigma = \sigma^\vee \cap M$

Each $m \in M_\sigma \subset M \longrightarrow M^Y$, gives a Cartier divisor $D_m$ on Star $Y$.

$$
\begin{tikzcd}
\operatorname{Spec} R \arrow[r, "\phi"] & X \\
u\pi^{\alpha} = \phi^* D_m \arrow[u, hook] \arrow[r] & D_m \arrow[u, hook]
\end{tikzcd}
$$

then $v\left(\phi^*(D_m)\right) \in \mathbb{Z}$

This give a map $M_\sigma \to \mathbb{Z}$, hence an element of $N_\sigma$.

If $R = R_v$, we denote this element $n_v$.

## Definition (morphism of toroidal embeddings):

Let $(U_X \subset X)$ and $(U_Z \subset Z)$ be toroidal embeddings. A dominant morphism $f : X \to Z$ is said to be toroidal if étale locally, near every closed point $x \in X$, there is a toric chart for $X$ near $x$, and a toric chart for $Z$ near $f(x)$ such that on these charts, $f$ become a toric map of toric varieties.

i.e. $\forall x \in X$ closed, there exists local models $(X_\sigma, s)$ at $x$, $(Z_\tau, t)$ at $f(x)$ and a toric morphism $g : X_\sigma \to Z_\tau$, so that the following diagram commutes

$$
\begin{tikzcd}
\hat{\mathcal{O}}_{X,x} & \hat{\mathcal{O}}_{X_\sigma,s} \arrow[l, "\simeq"'] \\
\hat{\mathcal{O}}_{Z,f(x)} \arrow[u, "\hat{f}^*"] & \mathcal{O}_{Z_\tau,t} \arrow[l, "\simeq"'] \arrow[u, "\hat{g}^*"']
\end{tikzcd}
$$

## Lemma:

A toroidal morphism $f : X \to Z$ induces a morphism $f_\Sigma : \Sigma_X \to \Sigma_Z$, which for each cone $\sigma' \in \Sigma_X$ is the restriction of $g_{\sigma\tau} : N^\sigma \to N^\tau$ where $(X_\sigma, s, N^\sigma)$ and $(Z_\tau, t, N^\tau)$ are local models at $x \in Y \subset X$, $f(x) \subset Y' \subset Z$, and $g_{\sigma\tau}$ linear map determined by the toric morphism $g : X_\sigma \to Z_\tau$.

In particular, $f_\Sigma(N^\sigma) \subset N^\tau$ for each $f_\Sigma(\sigma) \subset \tau$.

Proof: what is not clear is if we take another $x' \in X$ st $f(x') \notin Z$ ....

The dual $f_\Sigma^\vee : M^Z \longrightarrow M^Y$ away from $U_Z$

Q: Same as pull back C.D on star Z ?

is defined by pulling back a Cartier divisor and restrict it to star $Y$: it gives a Cartier divisor defined away from $U_X$. But the pull-back doesn't depend of the strata, so $f_\Sigma$ is well defined. ∎

## Theorem (Toroidalizing a morphism): Abramovich-Karu

Let $f : X \to Y$ be a dominant morphism of varieties. Then there exists modifications $X' \to X$ and $Y' \to Y$ and toroidal structures $U_{X'} \subset X'$, $U_{Y'} \subset Y'$ st the resulting rational map $f' : X' \to Y'$ is a toroidal morphism:

$$\begin{tikzcd} U_{X'} \arrow[r, hook] \arrow[d] & X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ U_{Y'} \arrow[r, hook] & Y' \arrow[r] & Y \end{tikzcd}$$

furthermore, $f'$ can be chosen flat.

## Definitions:

### • Toroidal firmament:

Let $(U \subset X)$ be a toroidal embedding with complex $\Sigma_X$. A toroidal firmament on $X$ is a finite collection $\Gamma = \{\Gamma_\sigma^i \subset N_\sigma\}$ with $\sigma \in \Sigma_X$ where.

- each $\Gamma_\sigma^i \subset \Gamma_\sigma$ is $fg$ submonoid
- each $\Gamma_\sigma^i$ generates $\sigma$ as a cone.
- the collection is closed under restrictions to faces, i.e for each $\Gamma_\sigma^i$, $\tau < \sigma$, $\exists j$ st $\Gamma_\sigma^i \cap \tau = \Gamma_\tau^j$.
- it is irredundant ($\Gamma_\sigma^i \not\subset \Gamma_\tau^j$ for $i, j$).

## • Morphism of toroidal firmaments:

a morphism from a toroidal firmament $\Gamma_X$ on $U_X \subset X$ to $\Gamma_Y$ on $U_Y \subset Y$ is a morphism $f : X \to Y$ with $f(U_X) \subset U_Y$ st $\forall \sigma$ in $\Sigma_X$, $\forall i$, if $f_\Sigma(\sigma) \subset \tau$, we have $f_\Sigma(\Gamma^i_\sigma) \subset \Gamma^j_\tau$ for some $j$.

## • Induced toroidal firmament:

We say that a toroidal firmament $\Gamma_X$ is induced by $f : X \to Y$ from $\Gamma_Y$ if $\forall \sigma \in \Sigma_X$ and $\tau \in \Sigma_Y$ st $f_\Sigma(\sigma) \subset \tau$ we have $\Gamma^i_\sigma = f_\Sigma^{-1}(\Gamma^j_\tau) \cap N_\sigma$ for some $j$.

• Given a proper birational equivalence $\phi : X_1 \dashrightarrow X_2$ then two toroidal firmaments $\Gamma_{X_1}$ and $\Gamma_{X_2}$ are said to be equivalent if there is a toroidal embedding $U_3 \subset X_3$ and a commutative diagram:

$$\begin{tikzcd}
 & X_3 \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
X_1 \arrow[rr, dashed, "\phi"'] & & X_2
\end{tikzcd}$$

where $f_i$ are modifications sending $U_3$ to $U_i$ and such that the two toroidal firmaments on $X_3$ induced by $f_1$ and $f_2$ from $\Gamma_{X_i}$ are identical.

firmament on any var
morphism of firmament

• A firmament on an arbitrary $X$ is the same as an equivalence class represented by a modification $X' \to X$ with $U' \subset X'$ a toroidal embedding and a firmament $\Gamma$ on $X'$.
A morphism of firmaments is a morphism of varieties which becomes a morphism of toroidal firmaments on some toroidal model.

• The trivial firmament is defined by $\Gamma^i_\sigma = N_\sigma$ $\forall \sigma \in \Sigma$.

# Definition (Base firmament):

1) $f: X \to Y$ a flat toroidal morphism of toroidal embeddings. The base firmament $\Gamma_f$ associated to $f$ is defined by the images

$$f_\tau(\Gamma_\sigma) := \Gamma_\sigma^\tau \quad \text{for each cone } \sigma \text{ over } \tau \qquad \underset{\in \Sigma_X}{f(\sigma)} \subset \underset{\in \Sigma_Y}{\tau}$$

2) Let $f: X \to Y$ a dominant morphism of varieties.

The base firmament of $f$ is represented by any $\Gamma_{f'}$ where $f': X' \to Y'$ is a flat toroidal model of $f$.
(Thus: does not depend of the model)

3) If $X$ reducible, $X = \bigcup X_i$ and $f: X_i \to Y$ dominant for each $i$, we define the base firmament by the (maximal elements) of the union of all firmaments associated to $X_i \to Y$.

Q: not clear if same toroidal structure on $Y$.

# Valuative firmament / equivalence of definitions:

Let $X$ be a variety, $X'$ a toroidal model and $\Gamma$ a firmament on $X'$. Let $v$ a valuation on $X$, we have $n_v \in N_\sigma$. (for some $\sigma$).

We define $\Gamma_v = \{ b \in \mathbb{N} \mid b\, n_v \in \Gamma_\sigma^i \text{ for som } i \}$.

This gives a firmament in the valuative definition.

Conversely, a firmament in the valuative definition has finitely many étale charts $Y_i \to Y$ where the firmament comes from $X_i \to Y_i$. One can toroidalize each $X_i \to Y_i$ simultaneously over some toroidal structure $U \subset Y$.

if $U \subset Y$ toroidal embedding, then $f_i^{-1}(U) \hookrightarrow Y_i$ is also one.

$$\begin{array}{ccccc} f_i^{-1}(U) \subset Y_i & \xrightarrow[f_i]{\text{étale}} & Y \supset U \\ x & \longmapsto & f_i(x) \\ \cup & & \cup \\ \text{Zariski } f_i^{-1}(Z_i) & & Z_i \text{ Zariski} \end{array}$$

$$f_i^{-1}(Z_i) \xrightarrow[\text{étale}]{f_i} Z_i \xrightarrow[\psi_i]{\text{étale}} X_\sigma \supset T \qquad \psi_i^{-1}(T) = U$$

$$f_i^{-1} \circ \psi_i^{-1}(T) = f_i^{-1}(U)$$

So "simoultaneously" means we consider the toroidal embedding $f_i^{-1}(U) \hookrightarrow Y_i$ $\forall i$ and toroidalize the $X_i \longrightarrow Y_i \xrightarrow{g_i} Y$

$$\begin{array}{ccc} Y_i & \xrightarrow{g_i} & Y \\ \cup & & \cup \\ f_i^{-1}(U) & & U \end{array}$$

The two procedures are inverse to each other. In particular, this shows that any firmament supports a unique constellation.

( firmament $\overset{1-1}{\longleftrightarrow}$ valuative firmament $\rightsquigarrow \Delta = \left(1 - \frac{1}{m_v}\right) v$ ), $\quad m_v := \min \Gamma_v$.

## Examples:

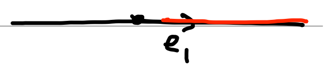
(1) $f : \mathbb{A}^2 \longrightarrow \mathbb{A}^1$ is a toric map of toric varieties

$k[x,y] \hookleftarrow k[t]$

$x^2 \hookleftarrow t$

Let's compute $\Gamma_f$.

$\mathbb{A}^1$ is an affine toric variety



$\sigma = \mathrm{cone}(e_1)$ $\qquad N_1 = \mathbb{Z}$ $\qquad M_1 = N^\vee \simeq \mathbb{Z}$

$N_\sigma = \sigma \cap N = \mathbb{N}$

$M_\sigma = \sigma^\vee \cap N = \mathbb{N}$ $\qquad 1 \mapsto t$

$\mathbb{A}^1 = \mathrm{Spec}\, k[M_\sigma] = \mathrm{Spec}\, k[\mathbb{N}]$ $(\simeq \mathrm{Spec}\, k[t])$

$\mathbb{A}^2$ is an affine toric variety

$\sigma' = \mathrm{cone}(e_1, e_2)$ $\qquad N_2 = \mathbb{Z}^2$, $\quad M_2 = \mathbb{Z}^2$

$N\sigma = \mathbb{N}^2$ $\qquad M\sigma = \mathbb{N}^2$ $\qquad (0,1) \mapsto y$, $(1,0) \mapsto x$

$\mathbb{A}^2 = \mathrm{Spec}\, k[M\sigma] = \mathrm{Spec}\, k[\mathbb{N}^2]$ $(\simeq \mathrm{Spec}\, k[x,y])$

$f : \mathbb{A}^2 \to \mathbb{A}^1 \rightsquigarrow f_\Sigma : \underset{N_2}{\overset{}{\mathbb{N}^2}} \longrightarrow \underset{N_1}{\mathbb{N}}$, $\quad \mathbb{N}^2 \subseteq N_2$, $\mathbb{N} \subseteq N_1$

$(a,b) \longmapsto 2a$

So $\Gamma_f := \{ f_\Sigma(\mathbb{N}^2) = 2\mathbb{N} \}$

$e_1^\perp = \{u \in e_1^* \text{ st } \langle u, e_1\rangle = 0\} = 0$
"$M = M^\vee$"
$\cong \mathbb{Z}$

Let $v$ be a valuation on $\mathbb{A}^1$.

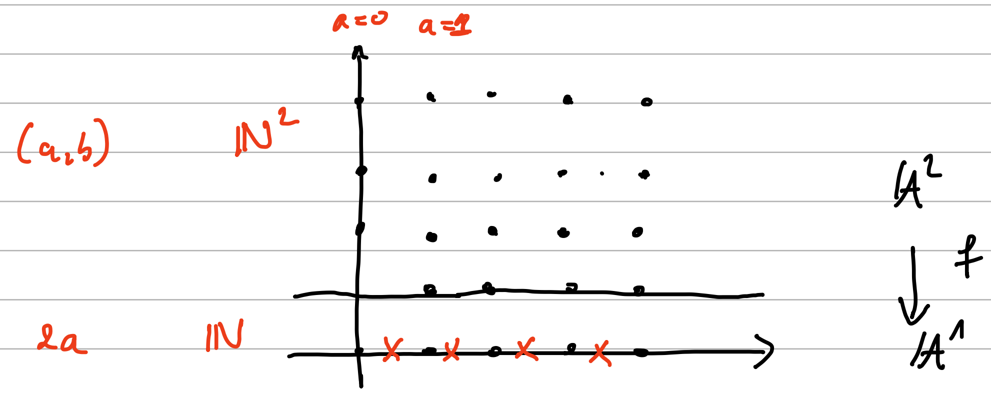
$f_n\, Q_{r,P_i}$ $P_i$

$$\begin{tikzcd} \mathrm{Spec}\, k[t]_{(t-a)} \arrow[r, "\phi"] & \mathbb{A}^1 = \mathrm{Spec}\, k[t] \\ \mathrm{Spec}\, k[t]_{(t-a)} \otimes_{k[t]} k[t]/t \simeq k[t]_{(t-a)}/(t) \arrow[u, hook] \arrow[r] & D = \{t=0\} = \mathrm{Spec}\, k[t]/t = \mathrm{Spec}\, k \arrow[u, hook] \end{tikzcd}$$

so $v(\phi^*(D)) = \begin{cases} 0 & \text{if } a \neq 0 \\ 1 & \text{otherwise.} \end{cases}$ ⟵ valuation of the monomials

if $a \neq 0$, $n_v = 0 \in \mathbb{N}$ ⇝ $\Gamma_v = \mathbb{N}$ ⇝ $m_v = 1$ ⇝ $\delta_v = 0$
if $a = 0$, $n_v = 1 \in \mathbb{N}$ ⇝ $\Gamma_v = \{k \in \mathbb{N} \mid k \in 2\mathbb{N}\}$
$m_v = \min \Gamma_v \setminus \{0\} = 2$ ⇝ $\delta_v = 1/2$

⇝ $\Delta_\Gamma = \frac{1}{2}(t=0)$.




2) $f : \mathbb{A}^2 \to \mathbb{A}^1$
$x^2 y \longleftarrow t$

$b_\mathbb{Z} : \mathbb{N}^2 \to \mathbb{N}$
$(a,b) \mapsto 2a+b$

So $\Gamma_b = \{ b_\mathbb{Z}(\mathbb{N}^2) = \mathbb{N} \}$

here $\Gamma_v = \{k \in \mathbb{N} \mid k n_v \in \mathbb{N}\}$ ⇝ empty condition!
$= \mathbb{N}$

⇝ $\Delta_\Gamma = 0$

4) $f : \mathbb{A}^2 \longrightarrow \mathbb{A}^1$
$x^2y^3 \longleftarrow\!\shortmid\; t$

$$f_\Sigma : \mathbb{N}^2 \longrightarrow \mathbb{N}$$
$$(a,b) \longmapsto 2a + 3b$$

$$\Gamma = \{ f(\mathbb{N}^2) = 2\mathbb{N} + 3\mathbb{N} \}$$

$$n_v = \begin{cases} 1 & \text{if} \quad a = 0 \qquad (t) \\ 0 & \text{if} \quad a \neq 0 \qquad (t-a) \end{cases}$$

$$\underset{a=0}{\Gamma_v} = \{ k \in \mathbb{N} \mid k\, n_v \in 2\mathbb{N} + 3\mathbb{N} \}$$
$$= \{ k \in \mathbb{N} \mid k \in 2\mathbb{N} + 3\mathbb{N} \}$$

$m_v = \min \Gamma_v = \overset{''}{k} = 2^* \quad \rightsquigarrow \delta_v = 1/2$

$\Delta_\Gamma = 1/2 \;\; (t=0)$ (Same constellation as ex 1)

6) $f : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$
$(x^2, y) \longleftarrow\!\shortmid\; (s,t)$

$$f_\Sigma : \mathbb{N}^2 \longrightarrow \mathbb{N}^2$$
$$(a,b) \longmapsto (2a, b)$$

$\sigma^\perp = \mathrm{Cone}(e_1, e_2)^\perp =$
$\{ m \in M_{\mathbb{R}} \mid \langle m, e_1 \rangle$

$$\Gamma_f = \{ f(\mathbb{N}^2) = 2\mathbb{N} \times \mathbb{N} \}$$

Let $v$ be divisorial valuation on $\mathbb{A}^2$.

$$\begin{tikzcd} \operatorname{Spec} R_v' \arrow[r, "\phi"] & \mathbb{A}^2 = \operatorname{Spec} k[s,t] \\ \operatorname{Spec} R_v'/s \arrow[u, hook] \arrow[r] & \begin{array}{c} D = \{s=0\} \\ D' = \{t=0\} \end{array} \arrow[u, hook] \end{tikzcd}$$

$$\left( \operatorname{Spec} R_v' = Y_v' \right.$$
$$\downarrow_1$$
$$Y$$
$$\downarrow$$
$$\left. \mathbb{A}^2 \right)$$

$m \in M \simeq \mathbb{Z}^2$
$\updownarrow \quad (1,0) \sim s$
$\quad (0,1) \sim t$

$$n_v := (v(s), v(t)) \in \mathbb{N}^2$$

$$\Gamma_v = \{ k \in \mathbb{N} \mid \underset{(k v(s),\, k v(t))}{k n_v} \in 2\mathbb{N} \times \mathbb{N} \}$$

so if $v(s) = 2a$, $m_v = 1 \rightsquigarrow \delta_v = 0$
if not $m_v = 2$, $\delta_v = 1/2$ $\rightsquigarrow$ $\Delta_\Gamma = \dots$

## Arithmetics:

**Definition:** Let $k$, $O_k$. $S \subset \operatorname{Spec} O_k$

An $S$-integral model of a toroidal firmament $\Gamma$ on $Y$ is an integral toroidal model $\mathcal{Y}'$ of $Y'$.

- Consider a toroidal firmament $\Gamma$ on $Y/k$ and $y \in Y(k)$. such that $\Gamma$ is trivial in a neighborhood of $y$.

$$\operatorname{Spec} k \xrightarrow{y} Y$$
$$* \longmapsto \text{strata} \rightsquigarrow \sigma$$

we want that $\Gamma^i \subseteq \underset{N\sigma}{\overset{\|}{N_\sigma}}$

Let $\mathcal{Y}$ be an $S$-integral model.

Then $y$ is a firm integral point of $\mathcal{Y}$ with respect to $\Gamma$ if the section $\operatorname{Spec} O_{k,S} \to \mathcal{Y}$ (exists?) is a morphism of firmaments, with $\operatorname{Spec} O_{k,S}$ given the trivial one.
toroidal embedding?

Explicitely, at each prime $p \in \operatorname{Spec} O_{k,S}$, where $y$ reduces to a stratum with cone $\sigma$, consider the associated point $n_{y_p} \in N\sigma$. Then $y$ is firmly $S$-integral if $\forall p$, $n_{y_p} \in \Gamma^i_\sigma$ for some $i$.

$$b_\tau : \mathbb{N} \to N\sigma$$

morp of firmaments $\iff b_\tau(\mathbb{N}) \subset \Gamma^i_\sigma$ $\quad$ $\operatorname{Spec} O_{k,S,p} \to Y$ (valuation)

$$\iff \underbrace{b_\tau(1)}_{= n_{y_p}} \in \Gamma^i_\sigma$$

$\mathrm{Spec}\, \mathcal{O}_{k,S,p} \longrightarrow \mathcal{Y}$

$s_p \longmapsto$ strata $H$ with cone $\sigma \rightsquigarrow N_\sigma$
$\sigma^{\circ}_H \qquad M_\sigma \to M^H$

$n_p$ is the "collection" of degree of tangencies of Cartier divisors on Star $H$ and $y$.

## Theorem :

$f : X \to Y$ proper dominant morphism of var$/k$.

There exists a toroidal birational model $X' \to Y'$ and an integral model $\mathcal{Y}'$ st the image of a rational point on $X'$ is a firm $S$-integ point on $\mathcal{Y}'$ with respect to $\Gamma_f$.

- Maybe after enlarging $S$, a point is firm $S$-integ on $\mathcal{Y}'$ with respect to $\Gamma_f$ iff locally in the étale top on $\mathcal{Y}'$, it lifts to a rational point on $X$.

## Conjecture

Let $(Y/\Delta)$ be a smooth proj Campana Constellation supported by a firmament $\Gamma$.
Then points on $Y$ f.integral with respect to $\Gamma$ are potentially dense iff $(Y/\Delta)$ is special.

$\hookrightarrow$ implies conj $*$.