

Table 6: Geometric Manin's Conjecture and examples

Goal :

80% - Give a precise statement of Geometric Manin's conjecture + examples validating the conjecture.

20% - §7 verifies GMC for Fano 3-folds of rank=1, index 2, describes the comp of $\text{Mor}(\mathbb{P}^1, X)$ by making computations in low degrees. Here we describe the deg=1 case.

Reminders :

X smooth uniruled proj variety, L big nef $\mathbb{Q}$ -divisor on X . (X weak Fano, $L := -K_X$)

- $a(X, L) = \min \{t \in \mathbb{R} : t[L] + [K_X] \in \overline{\text{Eff}}^1(X)\}$ $a(X, -K_X) = 1$
 - $F(X, L) = \{\alpha \in \text{Nef}_1(X) : (K_X + \alpha(X, L)L) \cdot \alpha = 0\}$ face of $\text{Nef}_1(X)$ $F(X, L) = \text{Nef}_1(X)$
 - $b(X, L) = \dim F(X, L)$ $b(X, L) = \dim \text{Nef}_1(X)$
 - $f: Y \rightarrow X$ is an a -cover if $\begin{cases} f \text{ dominant and then} \\ \text{proj. var.} \end{cases} \begin{cases} a(Y, f^*L) = a(X, L) \end{cases}$
- $f_*: \text{Nef}_1(Y) \rightarrow \text{Nef}_1(X)$
 $\downarrow$
 $F(Y, f^*L) \rightarrow F(X, L)$
- f is face contracting if $\begin{cases} f \text{ is an } a\text{-cover} \\ f_*: F(Y, f^*L) \rightarrow F(X, L) \text{ is not injective} \end{cases}$

Definition : To (X, L) , we associate $\alpha(X, L) \in \mathbb{R}_{>0}$ that will appear in the counting function of the components of $\text{Mor}(\mathbb{P}^1, X)$.

- We define $r(X, L)$ to be the smallest positive rational number of the form $L \cdot \alpha$, $\alpha \in N_1(X)_{\mathbb{Z}}$.
 $\hookrightarrow$ discrete set because L is $\mathbb{Q}$ -Cartier.
- We let V be the subspace of $N_1(X)$ spanned by $F(X, L)$. $V = N_1(X)$
 (this is a rational subspace wrt the lattice $N_1(X)_{\mathbb{Z}}$ of the curve class)
- We let $Q_r := \{\beta \in V \mid L \cdot \beta = r(X, L)\}$
 $\alpha(X, -K_X) = \text{Vol}(Q_r)$
 $\alpha(X, L) := \text{Volume of the polytope } Q_r \cap F(X, L)$ ✓
 $\hookrightarrow$ wrt a measure we define on Q_r

§1. Statement of the conjecture in the rigid case:

$$K(K_X + a(X, L)L) = 0$$

(eg. X weak Fano, $L = -K_X$, 0)

Manin's conjecture predicts the growth rate of components of $\text{Mor}(\mathbb{P}^1, X)$ after removing an "exceptional set".

In the number theory setting, removing points from a closed subset is not sufficient to obtain the expected growth rate, one needs to remove a thin set. Here, we will interpret a "thin set of rational curves" by the geometry of the invariants a and b .

More precisely, we will define which components of $\text{Mor}(\mathbb{P}^1, X)$ we take into account in our counting.

Definition 6.2: (Manin Component)

X sm proj unruled var, L big and nef $\mathbb{Q}$ -divisor on X with $K(K_X + a(X, L)L) = 0$.

Let $M \subset \text{Mor}(\mathbb{P}^1, X)$ be a component and let $\mathcal{C}$ be the universal family over M with

$s: \mathcal{C} \rightarrow X$ the evaluation map. We say that M is a Manin component if:

$$(h: \mathbb{P}^1 \rightarrow X, t \in \mathbb{P}^1) \mapsto h(t)$$

(i) The curves parametrized by M have class contained $F(X, L)$.

(ii) The morphism s doesn't factor rationally through any thin morphism $f: Y \rightarrow X$ st $a(Y, f^*L) > a(X, L)$.

This assumption is motivated by Thm 4.4: it says that $\text{Mor}_U(\mathbb{P}^1, X)$ has the expected dimension where

$$U := X \setminus \bigcup_{\text{subvarieties}} \{Y \subset X \text{ st } a(Y, f^*L) > a(X, L)\}$$

(iii) The morphism s doesn't factor rationally through any dominant thin morphism $f: Y \rightarrow X$ st f is face contracting and

$$(a(Y, f^*L), b(Y, f^*L)) \not\geq (a(X, L), b(X, L))$$

This is motivated by Conj 5.4

ignore contrib with higher a, b -val

(iv) S doesn't factor rati. through any dominant thin $f: Y \rightarrow X$ st
 $a(Y, f^*L) = a(X, L)$ and $K(K_Y + a(Y, f^*L) f^*L) > 0$.

This assumption is needed to get "uniqueness" is Manin Conj. + only consider v for
Remark: By Thm 4.4, Manin Components parametrize a dominant
family of curves.

Conjecture 6.5 (Strong Manin's Conjecture)

X sm proj uniruled var, L big nef $\mathbb{Q}$ -divisor with $K(K_X + a(X, L)L) = 0$.

For any $\mathbb{Z}$ -curve class α contained in the relative interior of $F(X, L)$,
there is at most one Manin component parametrizing curves of class α .

strong $\Rightarrow$ uniqueness

• To obtain the correct growth rate, it would be enough to show that the number of
Manin components representing an num class is bounded above, but uniqueness
holds in every example we know.

• Next, we formulate a weaker version which establishes the link with the
theory of rational points. Defining the counting function:

$$N(X, L, q, d) = \sum_{i=1}^d \sum_{W \in \text{Manin}_i} q^{\dim W}$$

where Manin_i is the set of Manin components of $\text{Mor}(\mathbb{P}^1, X)$ parametrizing
curves of L -degree $\underbrace{d r(X, -K_X)}_{\text{min } L \cdot \alpha}$.

Counts the number of Manin components of $\text{Mor}(\mathbb{P}^1, X)$
which param. curves of degree at most $d r(X, -K_X)$.

Conj 6.5:

$$\begin{aligned} N(X, L, q, d) &= \sum_{i=1}^d \sum_{\substack{\alpha \text{-}\mathbb{Z}\text{-curve} \\ \text{class}}} \sum_{W \in \text{Manin}_i} q^{\dim W} \\ &= \sum_{i=1}^d \sum_{\substack{\alpha \\ \text{Manin}_\alpha \neq \emptyset}} q^{\dim \text{Manin}_\alpha} \end{aligned}$$

(either 0 or 1.)

we give an
 $\rightarrow$ asymptotic of N

Conjecture 6.6: (Geometric Manin's conjecture)

X sm proj uniruled var. of $\dim = n$ and L big nef $\mathbb{Q}$ -divisor on X st $\kappa(K_X + a(X, L)L) = 0$. Then

$$N(X, L, q, d) \sim \frac{q^n d(X, L)}{1 - q^{-a(X, L)r(X, L)}} \underbrace{q^{da(X, L)r(X, L)} d^{b(X, L)-1}}_{B \log B'}$$

Remark: [LIT7, Thm 3.5] shows that there is a birational model $\phi: X \dashrightarrow X'$ st ϕ is a rational contraction, X' is normal $\mathbb{Q}$ -factorial with terminal singularities, and the anticanonical divisor on X' is big and nef and $-K_{X'} \equiv \phi_* a(X, L)L$. We expect Manin components to correspond to components of $\text{Mor}(\mathbb{P}^1, X')$.

So we are actually counting curves on some variety and not curves on some face.

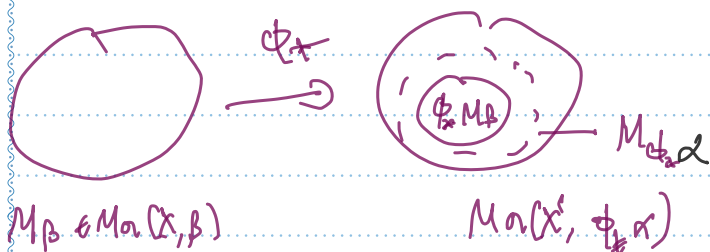
Heuristic argument:

$$\begin{array}{ccc} & F(X, L) & \\ & \downarrow & \\ \phi: X \dashrightarrow X' & \rightarrow & \text{Nef}_1(X) \xrightarrow{\phi_*} \text{Nef}_1(X') \\ & & \alpha \mapsto \phi_* \alpha \end{array}$$

$$\text{let } \beta \in \text{Nef}_1(X) \setminus F(X, L) \text{ st } \phi_* \beta = \phi_* \alpha \text{ for } \alpha \in F(X, L)$$

(assumptions $\Rightarrow \downarrow \rightarrow$ surjective)

$$\begin{array}{ccc} \text{Mor}(\mathbb{P}^1, X, \beta) & \xrightarrow{\phi_*} & \text{Mor}(\mathbb{P}^1, X', \phi_* \alpha) \\ \downarrow & & \downarrow \\ \text{Expdim } M_\beta & < & \text{Expdim } M_{\phi_* \alpha} \end{array} \quad \begin{array}{l} \text{mod comp} \\ \supset \phi_* M_\beta \end{array}$$



$$\bigcup_{\alpha \in \text{F}(X, Y)} \text{Mor}(\mathbb{P}^1, X) \neq \text{Man}(\mathbb{P}^1, X) = \bigcup_{\alpha \in \text{F}(X, Y)} \text{Mor}(\mathbb{P}^1, X, \alpha) \cup \phi_X M_\beta \cup \dots$$

for $\delta = \phi_X$

- This heuristic anticipates the existence of many families of rational curves with vanishing intersection against $K_X + a(X, L)L$. But the existence of even a single family is an open problem in birational geometry. (conj 6.13)

Conjecture 6.5 is known for the following Fano varieties:

- General hypersurfaces in $\mathbb{P}^n$ of $\deg < n-1$. (E. Riedl & D. Yang)
- Homogeneous varieties (J.F. Thomsen & B. Kim & R. Pandharipande)
- Del Pezzo surfaces (D. Testa)
- Fano toric (D. Bouquie)

Example 6.9 (Fano toric)

Let X be smooth proj toric variety with open torus U . Bouquie showed that every nef curve class which has intersection ≥ 1 against every torus-invariant divisor is represented by a unique dominant family of rational curves. By studying a and b invariants subvarieties and covers of toric varieties, L & T verify that the unique component representing a nef class with $\text{int} \geq 1$ is a Manin component, validating Conj 6.5.

§ 6.2 Manin's Conjecture: General Case

X sm proj uniruled var, L big and nef $\mathbb{Q}$ -divisor st $K_X + a(X, L)L > 0$. In this case, it induces an Iitaka fibration $\pi: X \rightarrow \mathbb{A}^1$ that we can assume to be a morphism (up to taking a birational model of X).

$$\underbrace{(K_X + a(X, L)L)_{\mathbb{Z}} = 0}_{\text{as a } \mathbb{Q}\text{-div on } X_{\mathbb{Z}} = \pi^{-1}(\mathbb{Z})} \quad \text{fiberwise, we are in the rigid case}$$

The fibers behave like the rigid case, so the definition of a Manin component needs to consider dominant families on X but also curves coming from the fibers of π . not only

- After this minor change, Conj 6.5 and the counting function $N(X, L, q, d)$ should be formulated exactly in a similar way.

§ 6.3 Manin-type bound

Theorem 6.11 : X sm proj uniruled var and L big nef $\mathbb{Q}$ -divisor on X . Fix $\epsilon > 0$.

Then for $q \gg 0$:

$$N(X, L, q, d) = O(q^{d(a(X, L) + \epsilon)})$$

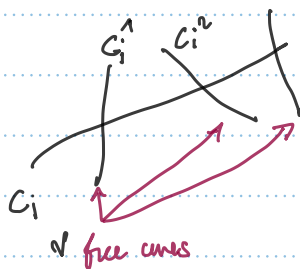
- It is conjectured that free curves generate the nef cone of curves. Z. Tian & H.R. Zong proved a weaker statement:

Lemma 6.12 : X sm proj rat. connected variety. Then $N_1(X)$ is spanned by the classes of free rational curves.

Proof : By Z. Tian and H.R. Zong

$N_1(X)$ is spanned by the classes of rational curves $\{C_i\}$

Since X is rationally connected, there is a family of very free curves C in X st through any point of X there is a very free curve from this family.



by gluing sufficiently many of these C_i^j with C_i , we can deform to get a free curve $\tilde{C}_i$

$$[C_i] + \underbrace{[C_i^1] + \dots + [C_i^k]}_{\text{free}} = [\tilde{C}_i]_{\text{free}}$$

Therefore, $N_1(X)$ is spanned by free curves $\square$

- Free curves that meet at a point can be glued to a free curve of a larger degree (by Kollar). Therefore one can generate many more dominant components of $\text{Mor}(\mathbb{P}^1, X)$ starting from the spanning set.

- If X is a Fano variety, $L = -K_X$. If all the components of rational curves constructed by gluing are Manin components, then we get a lower bound:

$$N(X, -K_X, q, d) \geq C q^{d \cdot \dim(X, -K_X)} d^{p(X)-1}$$

However, there is no reason for this in general.

§ 6.4. Geometric heuristic

In the statement of the Geometric Manin's conjecture, we said we should ignore contributions of $f: Y \rightarrow X$ with higher a, b values. Here we give a heuristic argument proving such components must be discounted.

Since we are giving heuristics, we assume the following difficult conjecture:

Conjecture 6.13 (unproved?)

X sm proj var k , L big and nef $\mathbb{Q}$ -divisor on X .

For each $\alpha \in \text{Nef}(X)_{\mathbb{Z}}$ st $(K_X + a(X, L)L) \cdot \alpha = 0$ with sufficiently high L -degree, there exists a dominant family of maps $P' \rightarrow X$ with class α .

Assuming this conj, the following two statements show that thin morphisms $f: Y \rightarrow X$ st Y has higher a, b -invariants would give contributions to the counting function which are higher than the predicted growth rate.

Proposition 6.14:

Assume Conj 6.13. Let X be a sm proj weak Fano variety and set $L = -K_X$. Suppose $f: Y \rightarrow X$ gen. finite with $a(Y, f^*L) > a(X, L)$. Then there exists components of $\text{Mor}(P', X)$ of any sufficiently high degree which factor through $f(Y) \subset X$ and have higher than the expected dimension.

Proof:

$$\text{Mor}(P', Y) \xrightarrow{f_*} \text{Mor}(P', X)$$

$$M_\beta \mapsto M_{f_*\beta}$$

$$\text{Exp dim } M_\beta - \text{Exp dim } M_{f_*\beta} = f^*L \cdot \beta \left(a(Y, f^*L) - a(X, L) + \dim Y - \dim X \right)$$

$$\text{Conj 6.13 on } (Y, f^*L), \text{ for } \beta \text{ st } f^*L \cdot \beta > \frac{\dim X - \dim Y}{a(Y, f^*L) - a(X, L)}$$

$\Rightarrow > 0$

$$\dim M_{f \rightarrow b} \geq \dim f_* M_\beta = \dim M_\beta \gg \text{Exp dim } M_\beta$$

f gen
finite

$$> \text{Exp dim } M_{f \rightarrow b}$$

□

Prop 6.15 : Assume Conj 6.13

X sm proj weak Fano variety and $L = -K_X$. Assume we have a surjective generically finite map $f: Y \rightarrow X$ which is a face contracting for L . There is a class $\alpha \in N_{\text{eff}}(X)_{\mathbb{R}}$ st the number of components of $\text{Mor}(\mathbb{P}^1, x, m\alpha)$ is bounded below by a polynomial of degree m equal to the relative dimension of the faces.

eg:
 f breaks
the wbc i.e.
 $b(Y, f^*L) >$
 $b(X, L)$

$\Rightarrow$ face
contract

(Idea) Proof

• There exists $\alpha \in N_1(X)$ st for m large enough -
there are $\geq C m^b$ pts $\beta \in F(Y, f^*L)$ st $f_* \beta = m\alpha \in N_1(X)$

$$\dim F(Y, f^*L) - F_{\text{only } Y \text{ LK}}$$

• Using Conj 6.13, we show that for m sufficiently large, for each such β , there is a component $M_\beta \subset \text{Mor}(\mathbb{P}^1, Y, \beta)$ parametrizing a dominant family of rational curves.

$$\begin{aligned} \text{Mor}(\mathbb{P}^1, Y, \beta) &\xrightarrow{f_*} \text{Mor}(\mathbb{P}^1, X, f_* \beta) \\ \text{vary } \beta &\leadsto \geq C m^b \text{ comp on } Y \xrightarrow{\text{to } X} \\ \text{at most deg } f &\neq \text{compts on } Y \longrightarrow \text{identified under } f_* \text{ to} \\ &\text{a single component in } X \end{aligned}$$

+ computation

$$C m^{b - \deg f + 1} \leq$$

□

Example 6.16

- Assuming Conj 6.13, Hanis, Roth and Starr show that a general hypersurface with deg not too large will not have subvarieties with higher a -values. We then should expect Manin's conj to hold for such hypersurfaces with no exceptional set (in the NT setting). Indeed such result is obtained by Birch using the circle method when the dimension is exponentially larger than its degree.
- Conversely, Browning and Vishe uses CM to prove statements about $\text{Mor}(\mathbb{P}^1, X)$ over $\mathbb{C}$ for hypersurface of low degree.

§ 7. Fano 3-folds of Picard rank 1 and index 2

X sm Fano 3-folds, (proj?) st $\text{Pic}(X) = \mathbb{Z}H$, $-K_X = 2H$ and $H^3 \geq 2$.

For such varieties, the behavior of a and w.r.t subvarieties is understood completely (by LT).

Using the theory worked out before, we can classify completely all components of $\text{Mor}(\mathbb{P}^1, X)$ after making few computations in low degree.

The main result will be Thm 7.9 which verifies Conj 6.5 for Fano 3-folds of this type (next talk).

(completes the list I gave at the start)

- We recall the classification of Fano 3-folds as above:

Thm 7.1 (Iskovskikh & Yu) X sm Fano 3-fold with $\text{Pic}(X) = \mathbb{Z}H$

$-K_X = 2H$ and $H^3 \geq 2$. Then we have $2 \leq H^3 \leq 5$ and X has the following description:

- when $H^3 = 5$, X is a section of the Grassmannian $G(1, 4)$ of lines in $\mathbb{P}^4$ by a general linear subspace of codim = 3.
- when $H^3 = 4$, X is a complete intersection of two quadrics in $\mathbb{P}^5$.

- When $H^3 = 3$, X is a cubic 3-fold in $\mathbb{P}^4$.
- $H^3 = 2$, X is a double cover of $\mathbb{P}^3$ ramified along a sm. quintic surface.

↳ We explain the geometric behavior of a and b .

Lemma 7.2: X sm 3-fold as above.

- There is no subvariety with $a(Y, -K_X|_Y) > a(X, -K_X)$.
- Let W denote the variety of lines on X and $\mathcal{U} \rightarrow W$ denote (moduli space par. curves in X of $\deg = 1$ against $L = -K_X$)

the universal family with $s: \mathcal{U} \rightarrow X$.

$$\text{Then } a(\mathcal{U}, -s^*K_X) = a(X, -K_X)$$

$$b(\mathcal{U}, -s^*K_X) = b(X, -K_X)$$

Furthermore, any dominant thin map $f: Y \rightarrow X$ st $a(Y, -f^*K_X) = a(X, -K_X)$

factors rat. through $\mathcal{U}$.

- Let α be a curve class on X st $H \cdot \alpha = d$

The computations above and § 6.1 suggest that $\text{Mor}(\mathbb{P}^1, X, \alpha)$ consists of two irreducible components:

- A Component R_d (Manin Comp) parametrizing dominant families of curves (rational to the image).

(Manin component excludes maps to X that factor via f of lemma 7.2, & that could factor through $\mathcal{U}$)

- A component N_d parametrizing morphisms which factor through $\mathcal{U}$.

- This is proved for cubic 3-folds by Stan

complete int of 2 quadrics by Cas trivet

The next goal is to verify this expectation for other Fano 3-folds with properties above -

Conj 6.6
at most
Conj 6.13
ds>0 exists

We start with low degree.

$\overline{M}_{0,0}(X,1)$: moduli space of stable maps

$C \rightarrow X$ with $g(C) = 0$

and $H \cdot [C] = 1$

Thm 7.3: X sm Fano 3-fold st $\text{Pic}(X) = \mathbb{Z}H$

$-K_X = 2H$

$H^3 \geq 2$

$\overline{M}_{0,0}(X,1)$ is isomorphic to the var of lines on X . In particular it is irred and gen-pair. a free curve

~> Not a Marini component (?)

(Next week: $\overline{M}_{0,0}(X,2)$ with details)