

Log Smoothness / stable curves

$M_g \subset \overline{M}_g$ moduli space of stable curves of genus g
 $\leadsto$ only allow nodes as singularities.

Question: What is great about nodes?

Answer: From the point of view of log geometry, these are logarithmically smooth. Indeed, there is a way to equip the schemes with log structure and promote a morphism $f: X \rightarrow Y$ of schemes to a morphism of log schemes $f: X \rightarrow Y$ that is log smooth.

§ 1. Formally Smooth morphisms of schemes:

Definition:

A ring map $R \rightarrow A$ is called formally smooth if for every commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B/I \\ \uparrow & \searrow & \uparrow \\ R & \longrightarrow & B \end{array}$$

where $I \subset B$ is an ideal such that $I^2 = 0$, there exists a lift $A \dashrightarrow B$ making everything commute.

(If A, B are R -algebras, $I \subset B$ st $I^2 = 0$ then $A \rightarrow B/I$
lifts to $A \rightarrow B$)

Definition:

$f: X \rightarrow S$ a morphism of schemes. It is said to be formally smooth if, given any commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\quad} & T = \operatorname{Spec} A/I \\ f \downarrow & \swarrow \text{dashed} & \downarrow i \\ S & \xleftarrow{\quad} & T' = \operatorname{Spec} A \end{array}$$

where $i: T = \operatorname{Spec} A/I \hookrightarrow T' = \operatorname{Spec} A$ a closed immersion st $I^2 = 0$, there exists a lift $T' \rightarrow X$.

Remark:

If $f: X \rightarrow S$ is a morphism of affine schemes, then f is formally smooth iff $\mathcal{O}_S(S) \rightarrow \mathcal{O}_X(X)$ is formally smooth map of rings.

Proposition:

$f: X \rightarrow S$ a morphism of schemes. The following are equivalent:

(1) f is formally smooth.

(2) f is smooth and locally of finite presentation

Example: Take S locally noetherian, $f: X \rightarrow S$ of finite type

then if f is smooth, it is formally smooth.

(Eg: $S = \operatorname{Spec} k$, $f: X = \mathbb{A}_k^n \rightarrow S = \operatorname{Spec} k$)

Reminder:

- $R \rightarrow R'$ of f t $\Rightarrow R' = R[x_1, \dots, x_n]/I$
- $R \rightarrow R'$ of f p $\Rightarrow R' = R[x_1, \dots, x_n]/I$ with I f -generated

Non-Example:

$$X = \text{Spec } k[x, y]_{/xy}$$

$$Y = \text{Spec } k$$

Then the natural map $X \rightarrow Y$ is not formally smooth. Indeed, take

$$T' = \text{Spec } k[\varepsilon]_{/\varepsilon^2}$$

$$T = \text{Spec } k[\varepsilon]_{/\varepsilon^3}.$$

$$\begin{aligned} (k[\varepsilon]_{/\varepsilon^3}) &\longrightarrow k[\varepsilon]_{/\varepsilon^2} \\ \begin{aligned} k[\varepsilon] &\ni \varepsilon \bmod \varepsilon^3 \mapsto \varepsilon \bmod \varepsilon^2 \\ (k[\varepsilon]_{/\varepsilon^3})_{/[\varepsilon^2]} &= k[\varepsilon]_{/\varepsilon^2} \end{aligned} \end{aligned}$$

Hence $T' \hookrightarrow T$ is a closed immersion with $(\varepsilon)^2 = 0$ in $k[\varepsilon]_{/\varepsilon^3}$.

Consider the tangent vector at the origin

$$(0, 0) \in X(k) = \{(a, b) \in k \mid ab = 0\}$$

This corresponds to a morphism

$$\text{Spec } k[x, y]_{/xy} \leftarrow \text{Spec } k[\varepsilon]_{/\varepsilon^2}$$

$$\begin{aligned} \text{given by } x &\mapsto \varepsilon \\ y &\mapsto \varepsilon \end{aligned}$$

$$(xy \mapsto \varepsilon^2 = 0 \quad : \text{ well-defined})$$

We have a commutative diagram

$$\begin{array}{ccc} \text{Spec } k[x, y] / \langle xy \rangle & \longleftarrow & \text{Spec } k[\epsilon] / \epsilon^2 \\ \downarrow & & \downarrow \\ \text{Spec } k & \longleftarrow & \text{Spec } k[\epsilon] / \epsilon^3 \end{array}$$

If there is a lifting $\text{Spec } k[\epsilon] / \epsilon^3 \rightarrow X$

then the map would be given by

$$\begin{aligned} x &\mapsto \epsilon + a\epsilon^2 \\ y &\mapsto \epsilon + b\epsilon^2 \end{aligned}$$

$$\text{But } xy \mapsto (\epsilon + a\epsilon^2)(\epsilon + b\epsilon^2) = \epsilon^2 \neq 0$$

Hence $X / \text{Spec } k$ is not formally smooth ($\Rightarrow$ not smooth)

§ 2. Log Smoothness:

In what follows, we add a new structure to our schemes (a log structure), define a smoothness that respects this structure (log smoothness), which make nodal curves

into $(\log)$ smooth morphisms.

§ 2.1. Log schemes :

Definition :

Let $\underline{X}$ be a scheme.

A log structure on $\underline{X}$ is the data of :

1) $M_{\underline{X}}$ a sheaf of monoids on $\underline{X}$.

2) A map $\alpha_x: M_x \rightarrow \mathcal{O}_x$ st $\alpha_x^{-1}(\mathcal{O}_x^\times) = \mathcal{O}_x^\times \subset M_x$
($M_x^\times = \mathcal{O}_x^\times$) prelog str.

Examples :

0) $M_X := \mathcal{O}_X^\times \hookrightarrow \mathcal{O}_X$ (trivial log str.)

1) $X = \text{Spec } k$ (a point).

How to make X into a log point?

$$M_X(X) := \mathbb{N} \oplus k^{\times} \xrightarrow{\text{monoid } X} \mathcal{O}_X(X) = k$$

$$(n, u) \mapsto \begin{cases} u & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases}$$

Ex: See that $\alpha_x^{-1}(k^\times) = k^\times$

it is called the standard log point.

2) $X = \text{Spec } k[t] = \mathbb{A}_k^1$

How to make X into a log line?

$$D = V(t) \quad U := X \setminus D$$

We define :

$$M_X(V) := \{ s \in \mathcal{O}_X(V) \mid s|_{V \cap U} \in \mathcal{O}_{V \cap U}^\times (V \cap U) \}$$

We have :

Ex: See that $M_X \xrightarrow{\alpha_X} \mathcal{O}_X$
 $\alpha_X^{-1}(\mathcal{O}_X^\times) = \mathcal{O}_X^\times$.

outside $u=0$, u is invertible


$$M_X(X) = \{ P \in k[u] \mid P \in k[u, u^{-1}]^\times \}$$

$$\Rightarrow \exists n \in \mathbb{N}, a \in k^\times \text{ st } P = a u^n$$

$$\Rightarrow M_X(X) = \mathbb{N} \oplus k^\times$$

$$\begin{aligned} M_X &\xrightarrow{\alpha_X} \mathcal{O}_X \\ (n, \chi) &\mapsto \chi u^n \end{aligned}$$

$$3) X = \text{Spec } k[u, y] = \mathbb{A}^2_k$$

How to make it into a log plane?

$$D = V(xy)$$

$$M_X \longrightarrow \mathcal{O}_X$$

$$\mathbb{N}^2 \oplus k^\times \longrightarrow k[u, y]$$

$$(a, b, \chi) \mapsto \chi x^a y^b$$

2 and 3 are called *divisorial log structures*.

$$4) X = \text{Spec } k[u, y]/xy$$

How to make X into a log curve?

$$M_X = \mathbb{N}^2 \oplus k^\times \longrightarrow k[u, y]/xy$$

$$(a, b, \chi) \mapsto \chi u^a y^b$$

Claim :

$(X = \text{Spec } k[x, y]/x, M_X) \rightarrow (Y = \text{Spec } k, M_Y)$
is log smooth.

Definitions : $f : (X, M_X) \rightarrow (Y, M_Y)$ a morphism of log schemes is a morphism of schemes $f : \underline{X} \rightarrow \underline{Y}$ and a commutative diag :

$$\begin{array}{ccc} M_X & \xrightarrow{\alpha_X} & \mathcal{O}_X \\ \uparrow & \curvearrowright & \uparrow \\ f^{-1}M_Y & \xrightarrow{f^{-1}\alpha_Y} & f^{-1}\mathcal{O}_Y \end{array}$$

• $f^{-1}M_Y \rightarrow \mathcal{O}_X$ prelog structure on $\underline{X}$.

↳ sheaf of monoids on $\underline{X}$

induces canonically a log structure on $\underline{X}$: the inverse image log structure.

$$\begin{array}{ccc} & & \rightarrow f^*M_Y \\ & & \downarrow \\ f^{-1}M_Y & \rightarrow & \mathcal{O}_X \end{array}$$

• $f : (X, M_X) \rightarrow (Y, M_Y)$ is said to be strict if $M_X = f^*M_Y$.

Remark :

$$(Sch/Y) \rightarrow (LSch/Y)$$

$$X \xrightarrow{f} Y \mapsto (X, f^*M_Y) \rightarrow (Y, M_Y)$$

Proof: is fully faithful functor

$$\begin{array}{ccc} f^* M_Y = M_X & \xrightarrow{\alpha_X} & \mathcal{O}_X \\ \uparrow & & \uparrow \\ f^* M_Y & \xrightarrow{f^* \alpha_Y} & f^* \mathcal{O}_Y \end{array}$$

$$\begin{array}{ccc} & \xrightarrow{\text{not strict}} & \\ T & \xrightarrow{\alpha} Z & \xrightarrow{\beta} Y \\ & \downarrow \text{strict} & \end{array}$$

$$M_T = h^* g^* M_Y = h^* M_Z$$

Definition:

$(X, M_X) \xrightarrow{f} (Y, M_Y)$ a morphism of log schemes. We say that f is an exact closed immersion if the underlying map of schemes $X \rightarrow Y$ is a closed immersion and $f^* M_Y = M_X$

Definition:

$(X, M_X) \xrightarrow{f} (Y, M_Y)$ a morphism of log schemes.

We say that f is formally smooth, if for any com.

diagram

$$\begin{array}{ccc} (X, M_X) & \longleftarrow & (T', L') \\ \downarrow f & & \downarrow i \\ (Y, M_Y) & \longleftarrow & (T, L) \end{array}$$

Not strict

with $T' = \text{Spec } R/I \xrightarrow{i} T = \text{Spec } R$ an exact closed immersion, with $I^2 = 0$ in R , there exists (étale) locally a lift $(X, M_X) \leftarrow (T, \mathcal{L})$ such that everything commutes.

We say that f is log smooth if it is formally smooth and of finite presentation.

Proposition:

A strict morphism of log schemes $(X, M_X) \xrightarrow{f} (Y, M_Y)$ is log smooth iff the underlying morphism of schemes is so.

Proof

- Assume f is f.log smooth + strict.

$$\begin{array}{ccc} \underline{X} & \xleftarrow{\quad} & \underline{T}' \\ \text{strict } f \downarrow & & \downarrow i \\ \underline{Y} & \xleftarrow{h} & \underline{T} \end{array}$$

$$\begin{array}{ccc} \leadsto \text{commutative!} & (\underline{X}, M_X = f^* M_Y) \xleftarrow{\quad} & (T', i^* \mathcal{O}_T) \\ & \downarrow & \downarrow \text{(strict)} \\ & (\underline{Y}, M_Y) \xleftarrow{\quad} & (\underline{T}, \mathcal{O}_T) \end{array}$$

By f.log smoothness, $\exists (\underline{T}, \mathcal{O}_T^*) \rightarrow (\underline{X}, M_X)$

making the diagram commute. In particular, a lift

$$\underline{T} \rightarrow \underline{X}$$

- Assume f is smooth + strict.

$$\begin{array}{ccc}
 (\underline{X}, f^* M_Y) & \xleftarrow{\quad} & (\underline{T}', i^* M_T) \\
 \downarrow f & \swarrow \text{---} \text{---} \text{---} \searrow h & \downarrow i \\
 (\underline{Y}, M_Y) & \xleftarrow{\quad} & (\underline{T}, M_T)
 \end{array}$$

since f is smooth, $\exists$ a lift $\underline{T} \rightarrow \underline{X}$.

$$\begin{array}{ccc}
 & M_T \rightarrow \mathcal{O}_T & \\
 & \downarrow & \downarrow \\
 \text{unique} & h^* M_X \rightarrow h^* \mathcal{O}_X & \\
 \text{choice} & \downarrow u & \\
 \text{of lift} & h^* f^* M_Y = (f \circ h)^* M_Y & \\
 \text{into} & & \\
 \text{a log map} & &
 \end{array}$$



Definition: R a ring.

Let P be a monoid, $\mathbb{1}$ the constant monoid.

$$\underline{P} \hookrightarrow R[P] = \mathcal{O}_{\text{Spec } R[P]}(\text{Spec } R[P]) \quad (\text{prelog})$$

induces a prelog structure on $\text{Spec } R[P]$

$\text{Spec } R[P] \leadsto$ there is a canonical associated log structure.

More generally, if (X, M_X) is a log scheme, and we have a morphism $\underline{P} \rightarrow M_X \rightarrow \mathcal{O}_X$; $\underline{P}$ is said to be a chart for the log str. if the

induced log structure on X is M_X .

This is equivalent to ask $X \rightarrow \text{Spec } \mathbb{C}[P]$
to be strict!

Morally, the chart is a monoid remembering
the essential information about the log structure.

Examples:

$$2) \quad \mathbb{N} \hookrightarrow \mathbb{Z}[\mathbb{N}]$$

The log structure induced by $\mathbb{N}$ on $A^1_{\mathbb{Z}}$ is
the divisorial log structure.

$$0) \quad \{0\} \rightarrow \mathcal{O}_X$$

$\{0\}$ is a $\overset{0}{\hookrightarrow} \overset{1}$ chart for the trivial log structure.

$$1) \quad \mathbb{N} \rightarrow \mathbb{k}$$
$$\begin{array}{ccc} 0 & \hookrightarrow & 1 \\ 1 & \hookrightarrow & 0 \end{array}$$

$\mathbb{N}$ is a chart for the standard log point

$$3) \quad \mathbb{N}^2 \rightarrow \mathbb{k}[x, y] / xy$$
$$(a, b) \mapsto x^a y^b$$

$\mathbb{N}^2$ is a chart.

Proposition :

Let P and Q be f.g (integral) monoids
 $Q \hookrightarrow P$ a monoid morphism, R a ring st
 $\ker(Q^{gp} \hookrightarrow P^{gp})$ and $\text{coker}(Q^{gp} \rightarrow P^{gp})$
are finite gps whose order is invertible
in R .

Let $X = \text{Spec } R[P]$, $Y = \text{Spec } R[Q]$
endowed with their canonical log structures.
 M_X, M_Y , respectively.

Then $(X, M_X) \rightarrow (Y, M_Y)$ is f log
smooth.

Theorem :

Let $f : (X, M_X) \rightarrow (Y, M_Y)$ be a morphism
of log schemes

Let $Q \rightarrow M_Y$ be a chart for (Y, M_Y) . Then the
following are equivalent :

(1) f is log smooth.

(2) (Etale) locally, there exists a chart

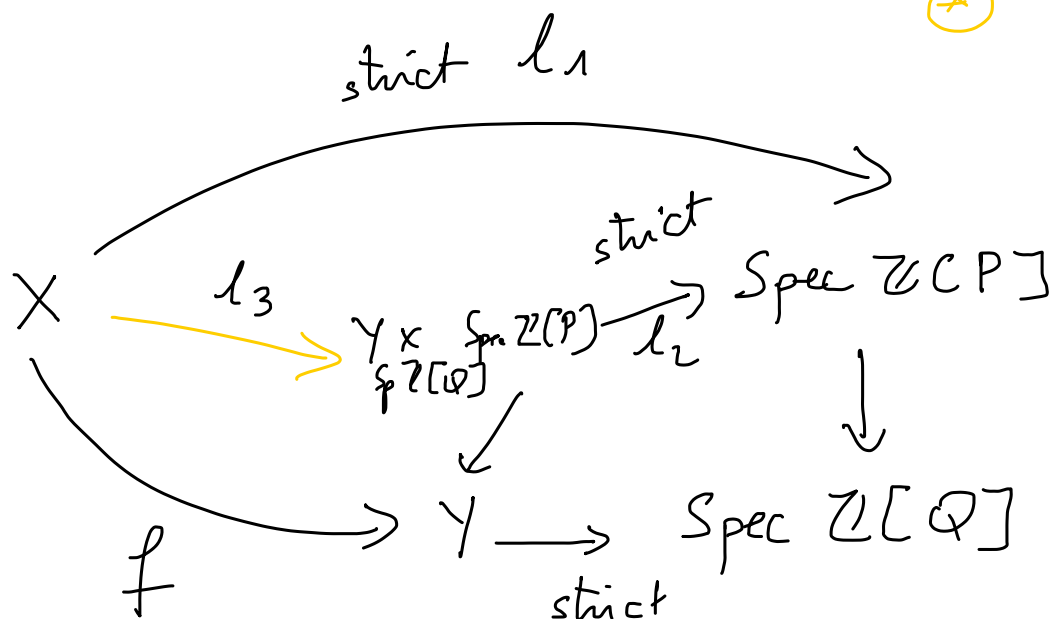
$P \rightarrow M_X$ for (X, M_X) , a morphism $Q \rightarrow P$ st

(a) the kernel / torsion ideal of $\mathbb{Q}^{gp} \rightarrow \mathbb{P}^{gp}$ are finite
 gps of order invertible in X .

(i.e. $\text{Spec } \mathbb{Z}[P] \rightarrow \text{Spec } \mathbb{Z}[Q]$ is log
 smooth)

(b) the induced morphism of schemes

(*)



$X \rightarrow Y \times_{\text{Spec } \mathbb{Z}[Q]} \text{Spec } \mathbb{Z}[P]$ is smooth.

(*)

$$\begin{aligned}
 M_X &= l_1^* M_{\text{Spec } \mathbb{Z}[P]} & l_1 \text{ strict} \\
 &= l_3^* l_2^* M_{\text{Spec } \mathbb{Z}[P]} & l_1 = l_2 \circ l_3 \\
 &= l_2^* M_{Y \times_{\text{Spec } \mathbb{Z}[Q]} \text{Spec } \mathbb{Z}[P]} & l_2 \text{ strict} \\
 &\Rightarrow l_3 \text{ strict}
 \end{aligned}$$

This theorem says the following :

A log smooth morphism of log schemes is the composition of a smooth morphism $(X \rightarrow Y)_{X \in \text{Spec } \mathbb{Z}[P], Y \in \text{Spec } \mathbb{Z}[Q]}$ and a homomorphism of monoids $Q \rightarrow P$ determining the log structure.

Example :

$$X = \text{Spec } k[x, y]/xy ; \quad Y = \text{Spec } k$$

Take M_X and M_Y to be the log structures on X and Y resp. induced by :

$$P = \mathbb{N}^2 \longrightarrow k[x, y]/xy$$

$$(a, b) \mapsto x^a y^b$$

$$Q = \mathbb{N} \longrightarrow k$$

$$\begin{array}{ccc} 0 & \mapsto & 1 \\ 1 & \mapsto & 0 \end{array}$$

$$a \mapsto 0^a$$

Let $f: (X, M_X) \rightarrow (Y, M_Y)$ the morphism:

$$\underline{X} = \text{Spec } k[t_1, y] / u_y \rightarrow \underline{Y} = \text{Spec } k$$

+ the commutative diagram:

$$\begin{array}{ccc} M_X \simeq \mathbb{N}^2 \oplus k^x & \xleftarrow{\Delta \oplus \text{id}} & \mathbb{N} \oplus k^x = M_Y \\ \alpha_X \downarrow & \searrow & \downarrow \alpha_Y \\ k[t_1, y] / u_y & \xleftarrow{\quad} & k \end{array}$$

$$\begin{array}{ccc} (n, n, u) & \xleftarrow{\quad} & (n, u) \\ \downarrow & & \downarrow \\ u \begin{smallmatrix} n & u \\ u & y \end{smallmatrix} & \xleftarrow{\quad} & u O^n \end{array}$$

$$\begin{cases} u & \text{if } n=0 \\ 0 & \text{otherwise} \end{cases} \quad (u_y = 0 \text{ in } k[t_1, y] / u_y)$$

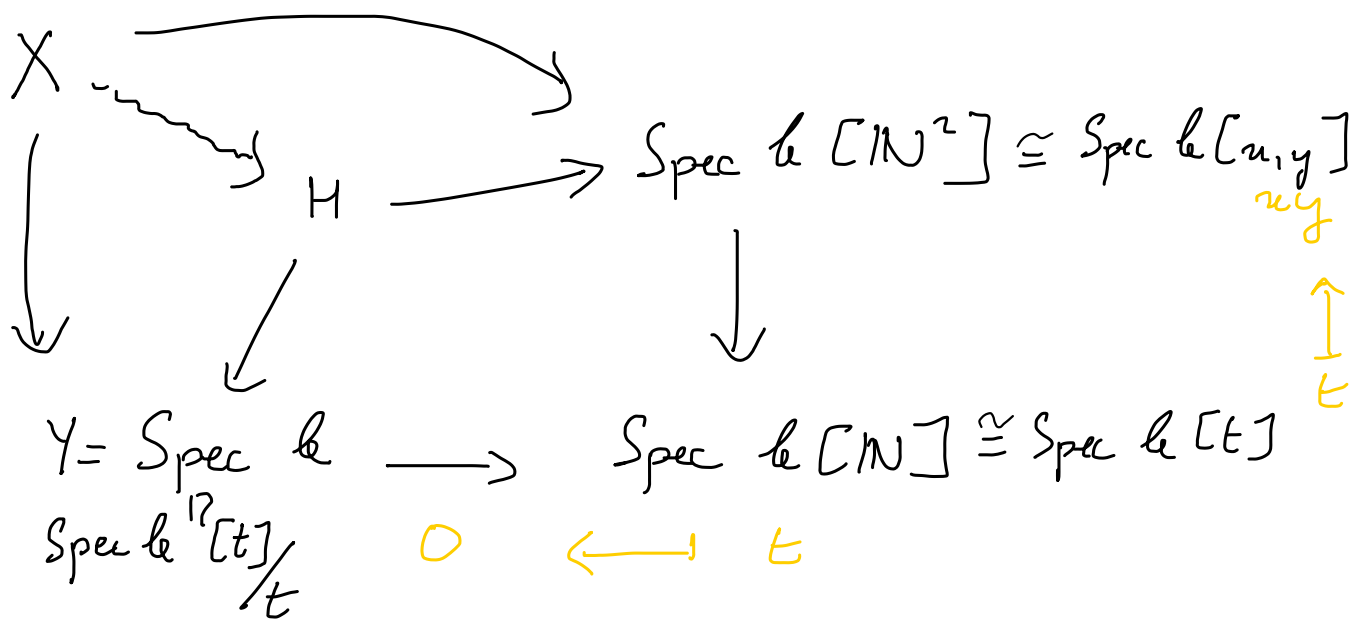
Let's check f is log smooth.

$$\begin{array}{ccc} a) & \mathbb{N} & \xrightarrow{\Delta} \mathbb{N}^2 \\ & \parallel & \parallel \\ & \mathbb{Q} & \mathbb{P} \\ & \Delta^{\text{gp}} & \\ & \mathbb{Z} & \xrightarrow{\quad} \mathbb{Z}^2 \quad a \mapsto (a, a) \end{array}$$

$\text{Ker } h = \{0\}$ finite of invertible order

$$\text{Tor } \underbrace{\text{Coker } \Delta^{\text{gp}}}_{\mathbb{Z}} = \{0\}$$

b)



$$\begin{aligned}
 k \otimes_{k[t]} k[u, y] &= k[t]_{/t} \otimes_{k[t]} k[u, y] \\
 &= k[u, y]_{/u_y}
 \end{aligned}$$

$$X = \text{Spec } k[u, y]_{/u_y} \longrightarrow H = \text{Spec } k[u, y]_{/u_y}$$

is the identity, so smooth.



