# Log Smoothness / Stable curves

$M_g \subset \overline{M}_g$ moduli space of stable curves of genus $g$

$\rightsquigarrow$ only allow nodes as singularities.

Question: What is great about nodes?

Answer: From the point of view of log geometry, these are logarithmically smooth. Indeed, there is a way to equip the schemes with log structures and promote a morphism $\underline{f}: \underline{X} \to \underline{Y}$ of schemes to a morphism of log schemes $f: X \to Y$ that is log smooth.

## § 1. Formally Smooth morphisms of schemes:

Definition:

A ring map $R \to A$ is called formally smooth if for every commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr, dashed] & B/I \\ R \arrow[u] \arrow[r] & B \arrow[u] \end{tikzcd}$$

where $I \subset B$ is an ideal such that $I^2 = 0$, there exists a lift $A \dashrightarrow B$ making everything commute.

(If $A, B$ are $R$-algebras, $I \subset B$ st $I^2 = 0$ then $A \to B/I$ lifts to $A \to B$)

## Definition:

$f: X \to S$ a morphism of schemes. It is said to be formally smooth if, given any commutative diagram

$$\begin{tikzcd} X & T = \operatorname{Spec} A/I \arrow[l] \arrow[d, hook, "i"] \\ S \arrow[u, leftarrow, "f"'] & T' = \operatorname{Spec} A \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

where $i: T = \operatorname{Spec} A/I \hookrightarrow T' = \operatorname{Spec} A$ a closed immersion st $I^2 = 0$, there exists a lift $T' \to X$.

## Remark:

If $f: X \to S$ is a morphism of affine schemes, then $f$ is formally smooth iff $\mathcal{O}_S(S) \to \mathcal{O}_X(X)$ is formally smooth map of rings.

## Proposition:

$f: X \to S$ a morphism of schemes. The following are equivalent:

(1) $f$ is formally smooth.

(2) $f$ is smooth and locally of finite presentation

**Example:** Take $S$ locally noetherian, $f: X \to S$ of finite type. Then if $f$ is smooth, it is formally smooth.

( Eg: $S = \operatorname{Spec} k$, $f: X = \mathbb{A}^n_k \to S = \operatorname{Spec} k$
$\mathbb{P}^n_k$

Reminder:

- $R \to R'$ of ft $\Rightarrow$ $R' = R[x_1, \dots, x_n]/I$
- $R \to R'$ of fp $\Rightarrow$ $R' = R[x_1, \dots, x_n]/I$ with $I$ f. generated

Non-Example :

$X = \operatorname{Spec} k[x,y]/xy$

$Y = \operatorname{Spec} k$

Then the natural map $X \to Y$ is not formally smooth. Indeed, take

$T' = \operatorname{Spec} k[\varepsilon]/\varepsilon^2$

$T = \operatorname{Spec} k[\varepsilon]/\varepsilon^3$.

$$\left(k[\varepsilon]/\varepsilon^3\right) \twoheadrightarrow k[\varepsilon]/\varepsilon^2$$

$[\varepsilon]$ : $\varepsilon \bmod \varepsilon^3 \mapsto \bar{\varepsilon} : \varepsilon \bmod \varepsilon^2$

$$\left(k[\varepsilon]/\varepsilon^3\right)/[\varepsilon^2] = k[\varepsilon]/\varepsilon^2$$

Hence $T' \hookrightarrow T$ is a closed immersion with $(\varepsilon)^2 = 0$ in $k[\varepsilon]/\varepsilon^3$.

Consider the tangent vector at the origin $(0,0) \in X(k) = \{(a,b) \in k \text{ st } ab = 0\}$

This corresponds to a morphism

$$\operatorname{Spec} k[x,y]/xy \leftarrow \operatorname{Spec} k[\varepsilon]/\varepsilon^2$$

given by $x \mapsto \varepsilon$

$y \mapsto \varepsilon$

$(xy \mapsto \varepsilon^2 = 0$ : well-defined$)$

We have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k[x,y]/xy \arrow[d] & \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \arrow[l] \arrow[d, hook] \\ \operatorname{Spec} k & \operatorname{Spec} k[\varepsilon]/\varepsilon^3 \arrow[l] \end{tikzcd}$$

If there is a lifting $\operatorname{Spec} k[\varepsilon]/\varepsilon^3 \to X$ then the map would be given by

$$x \mapsto \varepsilon + a\varepsilon^2$$
$$y \mapsto \varepsilon + b\varepsilon^2$$

But $xy \mapsto (\varepsilon + a\varepsilon^2)(\varepsilon + b\varepsilon^2) = \varepsilon^2 \neq 0$

Hence $X/\operatorname{Spec} k$ is not formally smooth ($\Rightarrow$ not smooth)

## §2. Log Smoothness:

In what follows, we add a new structure to our schemes (a log structure), define a smoothness that respects this structure (log smoothness), which make nodal curves into (log) smooth morphisms.

## §2.1. Log schemes:

**Definition:**

Let $X$ be a scheme.

A log structure on $X$ is the data of:

1) $M_X$ a sheaf of monoids on $X$.

2) $\underbrace{\text{A map } \alpha_X : M_X \to O_X}_{\text{Prelog stru.}}$ st $\alpha_X^{-1}(O_X^\times) = O_X^\times \subset M_X$

$(M_X^\times = O_X^\times)$

**Examples:**

0) $M_X := O_X^\times \hookrightarrow O_X$ (trivial log stru.)

1) $X = \operatorname{Spec} k$ (a point).

How to make $X$ into a log point?

$$M_X(X) := \mathbb{N} \oplus \underset{\text{monoid}\times}{k^\times} \longrightarrow O_X(X) = k$$

$$(n,u) \longmapsto \begin{cases} u & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Ex: See that $\alpha_X^{-1}(k^\times) = k^\times$

it is called the standard log point.

2) $X = \operatorname{Spec} k[x] = \mathbb{A}^1_k$

How to make $X$ into a log line?

$D = V(x)$ $\quad U := X \setminus D$

We define:

$$M_X(V) := \{ s \in O_X(V) \mid s|_{V \cap U} \in O_{V\cap U}^\times (V \cap U) \}$$

We have :

$$M_X \overset{\alpha_X}{\hookrightarrow} \mathcal{O}_X$$

Ex: See that $\alpha_X^{-1}(\mathcal{O}_X^*) = \mathcal{O}_X^*$.

outside $x=0$, $x$ is invertible

$$M_X(X) = \{ P \in k[x] \mid P \in k[x, x^{-1}]^\times \}$$

$$\Rightarrow \exists\, n \in \mathbb{N},\ \alpha \in k^\times \text{ st } P = \alpha x^n$$

$$\Rightarrow M_X(X) = \mathbb{N} \oplus k^\times$$

$$M_X \xrightarrow{\alpha_X} \mathcal{O}_X$$

$$(n, u) \longmapsto u x^n$$

3) $X = \operatorname{Spec} k[x, y] = \mathbb{A}^2_k$

How to make it into a log plane?

$D = V(xy)$

$$M_X \longrightarrow \mathcal{O}_X$$

$$\mathbb{N}^2 \oplus k^\times \longrightarrow k[x, y]$$

$$(a, b, u) \longmapsto u x^a y^b$$

2 and 3 are called divisorial log structures.

4) $X = \operatorname{Spec} k[x, y]/xy$

How to make $X$ into a log curve?

$$M_X = \mathbb{N}^2 \oplus k^\times \longrightarrow k[x, y]/xy$$

$$(a, b, u) \longmapsto u x^a y^b$$

Claim :

$(X = \operatorname{Spec} k[x,y]/_{xy},\ M_X) \to (Y = \operatorname{Spec} k,\ M_Y)$
is log smooth.

Definitions : $f : (X, M_X) \to (Y, M_Y)$ a morphism of log schemes is a morphism of schemes $\underline{f} : \underline{X} \to \underline{Y}$ and a commutative diag :

$$\begin{tikzcd} M_X \arrow[r, "\alpha_X"] & \mathcal{O}_X \\ f^{-1} M_Y \arrow[u] \arrow[r, "f^{-1}\alpha_Y"'] \arrow[ur, phantom, "\circlearrowright"] & f^{-1}\mathcal{O}_Y \arrow[u] \end{tikzcd}$$

- $f^{-1} M_Y \longrightarrow \mathcal{O}_X$ prelog structure on $\underline{X}$.
  $\hookrightarrow$ sheaf of monoids on $\underline{X}$
  induces canonically a log structure on $\underline{X}$ : the inverse image log structure.
  $$\begin{tikzcd} & f^* M_Y \arrow[d] \\ f^{-1} M_Y \arrow[ur] \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

- $f : (X, M_X) \to (Y, M_Y)$ is said to be strict if $M_X = f^* M_Y$.

Remark :

$$(Sch / Y) \longrightarrow (LSch / Y)$$
$$X \xrightarrow{f} Y \longmapsto (X, f^* M_Y) \to (Y, M_Y)$$

Proof: is fully faithfull functor

$$\begin{array}{ccc} f^*M_Y = M_X & \xrightarrow{\alpha_X} & \mathcal{O}_X \\ \uparrow & & \uparrow \\ f^{-1}M_Y & \xrightarrow[f^{-1}\alpha_Y]{} & f^{-1}\mathcal{O}_Y \end{array}$$

$$\begin{tikzcd} T \arrow[r, "h"] \arrow[rr, bend left, "n\ \text{strict}"] & Z \arrow[r, "g"'] & Y \end{tikzcd}$$

$g$ strict, $h$ strict

$M_T = h^*g^*M_Y$
$= h^*M_Z$

## Definition:

$(X, M_X) \xrightarrow{f} (Y, M_Y)$ a morphism of log schemes. We say that $f$ is an exact closed immersion if the underlying map of schemes $X \to Y$ is a closed immersion and $f^*M_Y = M_X$

## Definition:

$(X, M_X) \xrightarrow{f} (Y, M_Y)$ a morphism of log schemes. We say that $f$ is formally smooth, if for any com. diagram

$$\begin{tikzcd} (X, M_X) \arrow[d, "f"'] & (T', L') \arrow[l] \arrow[d, hook, "i"] \\ (Y, M_Y) & (T, L) \arrow[l] \end{tikzcd}$$

Not strict

with $T' = \mathrm{Spec}\, R/I \overset{i}{\hookrightarrow} T = \mathrm{Spec}\, R$ an exact closed immersion, with $I^2 = 0$ in $R$, there exists (étale) locally a lift $(X, M_X) \leftarrow (T, L)$ such that everything commutes.

We say that $f$ is log smooth if it is formally smooth and of finite presentation.

Proposition:

A strict morphism of log schemes $(X, M_X) \xrightarrow{f} (Y, M_Y)$ is log smooth iff the underlying morphism of schemes is so.

Proof

- Assume $f$ is f. log smooth + strict.

$$\begin{tikzcd} \underline{X} \arrow[d, "\underline{f}"'] & \underline{T}' \arrow[l] \arrow[d, hook, "i"] \\ \underline{Y} & \underline{T} \arrow[l, "h"'] \end{tikzcd}$$

strict

$\rightsquigarrow$ commutative!

$$\begin{tikzcd} (\underline{X}, M_X = f^* M_Y) \arrow[d] & (T', i^* \mathcal{O}_T) \arrow[l] \arrow[d, hook] \\ (\underline{Y}, M_Y) & (\underline{T}, \mathcal{O}_T) \arrow[l] \end{tikzcd}$$

(strict)

By f log smoothness, $\exists\ (\underline{T}, \mathcal{O}_T^*) \longrightarrow (\underline{X}, M_X)$ making the diagram commute. In particular, a lift $\underline{T} \longrightarrow \underline{X}$

- Assume $\underline{f}$ is smooth + strict.

$$\begin{tikzcd}
(\underline{X}, f^* M_Y) \arrow[d, "f"'] & & (\underline{T}', i^* M_T) \arrow[ll] \arrow[d, hook, "i"] \\
(\underline{Y}, M_Y) & & (\underline{T}, M_T) \arrow[ll] \arrow[ull, dashed, "h"']
\end{tikzcd}$$

since $\underline{f}$ is smooth, $\exists$ a lift $\underline{T} \to \underline{X}$.

unique choice of lift into a log map

$$\begin{tikzcd}
M_T \arrow[r] \arrow[d, dashed] & \mathcal{O}_T \arrow[d] \\
h^* M_X \arrow[r] & h^* \mathcal{O}_X
\end{tikzcd}$$

$$h^* M_X \supseteq h^* f^* M_Y = (f \circ h)^* M_Y$$

<u>Definition</u>: $R$ a ring.
Let $P$ be a monoid, $\underline{P}$ the constant monoid.
$\underline{P} \hookrightarrow R[P] = \mathcal{O}_{\mathrm{Spec}\, R[P]}(\mathrm{Spec}\, R[P])$ (prelog)
induces a prelog structure on $\mathrm{Spec}\, R[P]$
$\mathrm{Spec}\, R[P] \rightsquigarrow$ there is a canonical associated log structure.
More generally, if $(X, M_X)$ is a log scheme, and we have a morphism $\underline{P} \to M_X \to \mathcal{O}_X$; $P$ is said to be a chart for the log stru. if the

induced log structure on $X$ is $\mathcal{M}_X$.
This is equivalent to ask $X \to \operatorname{Spec} R[P]$ to be strict!

Morally, the chart is a monoid remembering the essential information about the log structure.

Examples:

2) $\mathbb{N} \hookrightarrow \mathbb{Z}[\mathbb{N}]$
The log structure induced by $\mathbb{N}$ on $\mathbb{A}^1_{\mathbb{Z}}$ is the divisorial log structure.

0) $$\begin{aligned} \{0\} &\longrightarrow \mathcal{O}_X \\ 0 &\longmapsto 1 \end{aligned}$$
$\{0\}$ is a chart for the trivial log structure.

1) $$\begin{aligned} \mathbb{N} &\longrightarrow k \\ 0 &\longmapsto 1 \\ 1 &\longmapsto 0 \end{aligned}$$
$\mathbb{N}$ is a chart for the standard log point

3) $$\begin{aligned} \mathbb{N}^2 &\longrightarrow k[x,y]/xy \\ (a,b) &\longmapsto x^a y^b \end{aligned}$$

$\mathbb{N}^2$ is a chart.

## Proposition :

Let $P$ and $Q$ be f.g (integral) monoids $Q \hookrightarrow P$ a monoid morphism, $R$ a ring st $\ker(Q^{gp} \hookrightarrow P^{gp})$ and (tor) coker $(Q^{gp} \to P^{gp})$ are finite gps whose order is invertible in $R$.

Let $X = \operatorname{Spec} R[P]$, $Y = \operatorname{Spec} R[Q]$ endowed with their canonical log structures. $M_X$, $M_Y$, respectively.

Then $(X, M_X) \to (Y, M_Y)$ is log smooth.

## Theorem :

Let $f : (X, M_X) \to (Y, M_Y)$ be a morphism of log schemes

Let $Q \to M_Y$ be a chart for $(Y, M_Y)$. Then the following are equivalent :

(1) $f$ is log smooth.

(2) (Etale) locally, there exists a chart $P \to M_X$ for $(X, M_X)$, a morphism $Q \to P$ st

(a) The kernal / torsion Cokernel of $Q^{gp} \to P^{gp}$ are finite gps of order invertible in $X$.

(i.e $\operatorname{Spec} \mathbb{Z}[P] \to \operatorname{Spec} \mathbb{Z}[Q]$ is log smooth)

(b) The induced morphism of schemes (*)

$$\begin{tikzcd}
X \arrow[rr, bend left, "\text{strict } l_1"] \arrow[r, "l_3"] \arrow[dr, bend right, "f"'] & Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P] \arrow[r, "\text{strict}", "l_2"'] \arrow[d] & \operatorname{Spec} \mathbb{Z}[P] \arrow[d] \\
 & Y \arrow[r, "\text{strict}"'] & \operatorname{Spec} \mathbb{Z}[Q]
\end{tikzcd}$$

$X \to Y \times_{\operatorname{Spec} \mathbb{Z}(Q)} \operatorname{Spec} \mathbb{Z}[P]$ is smooth.

(*)
$$\left\{\begin{aligned}
M_X &= l_1^* M_{\operatorname{Spec} \mathbb{Z}[P]} && l_1 \text{ strict} \\
&= l_3^* l_2^* M_{\operatorname{Spec} \mathbb{Z}[P]} && l_1 = l_2 \circ l_3 \\
&= l_3^* M_{Y \times_{\operatorname{Sp} \mathbb{Z}(Q)} \operatorname{Sp} \mathbb{Z}(P)} && l_2 \text{ strict} \\
&\Rightarrow l_3 \text{ strict}
\end{aligned}\right.$$

This theorem says the following:

A log smooth morphism of log schemes is the composition of a smooth morphism $(X \to Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P])$ and a homomorphism of monoids $Q \to P$ determining the log structure.

## Example:

$X = \operatorname{Spec} k[x,y]/_{xy}$ ; $Y = \operatorname{Spec} k$

Take $M_X$ and $M_Y$ to be the log structures on $X$ and $Y$ resp. induced by:

$$P = \mathbb{N}^2 \longrightarrow k[x,y]/_{xy}$$
$$(a,b) \longmapsto x^a y^b$$

$$Q = \mathbb{N} \longrightarrow k$$
$$0 \longmapsto 1 \qquad a \longmapsto 0^a$$
$$1 \longmapsto 0$$

Let $f : (X, M_X) \longrightarrow (Y, M_Y)$ the morphism :

$\underline{X} = \operatorname{Spec} k[x,y]/xy \longrightarrow \underline{Y} = \operatorname{Spec} k$

+ the commutative diagram :

$$\begin{tikzcd}
M_X = \mathbb{N}^2 \oplus k^\times \arrow[d, "\alpha_X"'] & \mathbb{N} \oplus k^\times = M_Y \arrow[l, "\Delta \oplus \mathrm{id}"'] \arrow[d, "\alpha_Y"] \arrow[dl, phantom, "\circlearrowright"] \\
k[x,y]/xy & k \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
(n,n,u) \arrow[d, mapsto] & (n,u) \arrow[l, mapsto] \arrow[d, mapsto] \\
u\, x^n y^n & u\, 0^n \arrow[l, mapsto]
\end{tikzcd}$$

$u\, x^n y^n = \begin{cases} u & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$ $(xy = 0 \text{ in } k[x,y]/xy)$

Let's check $f$ is log smooth.

a) $\underset{Q}{\mathbb{N}} \xrightarrow{\Delta} \underset{P}{\mathbb{N}^2}$

$\mathbb{Z} \xrightarrow{\Delta^{gp}} \mathbb{Z}^2 \qquad a \longmapsto (a,a)$

$\operatorname{Ker} h = \{0\}$ finite of invertible order

$\operatorname{Tor} \underbrace{\operatorname{Coker} \Delta^{gp}}_{= \mathbb{Z}} = \{0\}$

b)

$$\begin{tikzcd}
X \arrow[dr, squiggly] \arrow[rr, bend left] \arrow[dd] & & \operatorname{Spec} k[\mathbb{N}^2] \cong \operatorname{Spec} k[x,y] \arrow[dd] \\
& H \arrow[ur] \arrow[dl] & \\
Y = \operatorname{Spec} k \arrow[rr] & & \operatorname{Spec} k[\mathbb{N}] \cong \operatorname{Spec} k[t]
\end{tikzcd}$$

$\operatorname{Spec} k^{!?}[t]/t$

$0 \longleftarrow\!\shortmid\; t$

$xy \;\shortmid\!\!\longleftarrow\; t$

$$k \underset{k[t]}{\otimes} k[x,y] = k[t]/t \underset{k[t]}{\otimes} k[x,y]$$

$$= k[x,y]/xy$$

$X = \operatorname{Spec} k[x,y]/xy \longrightarrow H = \operatorname{Spec} k[x,y]/xy$

is the identity, so smooth.