Extending torsors via logarithmic schemes

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Motivation

Let X be a noetherian integral regular scheme, G a finite flat commutative X-group scheme and U a dense open subset of X whose complementary is a divisor D. Then the morphism of restriction

$$H^1_{fppf}(X,G) \to H^1_{fppf}(U,G)$$

is injective.

We would like to understand its image. In other words, given a G-torsor over U, we want to know under which conditions it admits an extension to a G-torsor over X.

Using *purity theorem* for torsors, one sees that this problem is of a local nature. Let $x \in D$ be a point of codimension 1, since X is regular, we know that $S := \mathrm{Spec}(O_{X,x})$ is a dvr. If we assume that G_S is an étale S-group scheme (this is the case in particular if the residual characteristic of x is prime to the order of G), then if a G-torsor over U extends over the point x, it is necessary for the corresponding extension of the generic point of S to be unramified. Such a situation is not common. For example, given a Galois extension of number fields, there is no reason for it to be unramified outside the places dividing the order of its Galois group.

In other words, asking for a G-torsor over U to extend into an fppf G-torsor over X is too restrictive if one wishes to understand the ramification on the complement of U. In addition, G being assumed to be flat, replacing the fppf topology by a finer topology does not create new representable torsors. Faced with this situation, it is appropriate to leave the world of fppf torsors and consider a larger category, namely that of Kummer log flat torsors.

Log torsors

The divisor D induces on X a fine and saturated log structure called the *divisorial* log structure; it is by definition trivial over U. Kummer log flat torsors over X are, roughly speaking, tamely ramified along D.

Example: A a discrete valuation ring with uniformizer π, $n \in A^\times$, $\zeta \in A$, $B := A[\sqrt[n]{\pi}]$, $G := \mathrm{Aut}_A(B) = \mu_n \simeq \mathbb{Z}/n\mathbb{Z}$, $X := \mathrm{Spec}(A)$, $Y := \mathrm{Spec}(B)$. Both schemes are endowed with their divisorial log structures induced by the residual fields, seen as divisors. Then:

- $Y \to X$ is not an fppf G-torsor because $A \to B$ is totally ramified, while G is unramified.
- $Y \to X$ is a Kummer log étale G-torsor.

Problem

Let R be a dvr with fraction field K and residue field k, C a smooth projective K-curve with a K-point Q, and G a finite commutative K-group scheme. Let $\mathcal{C}$ be an R-regular model of C endowed with the divisorial log structure induced by its special fiber (seen as a divisor) and let $\mathcal{Q}$ the R-log section extending Q. Let $Y \to C$ be an fppf pointed G-torsor (relatively to Q). When does it extend into a pointed log torsor over $\mathcal{C}$ (relatively to $\mathcal{Q}$)?

Main results

Weil: the Jacobian $\mathrm{Pic}^0_{C/K} = J$ of C classifies G-torsors over it:

$$H^1_{fppf}(C,Q,G) \simeq \mathrm{Hom}(G^D, J). \qquad (*)$$

Theorem (Raynaud): If $\mathcal{G}$ is a finite flat commutative R-group scheme, we have isomorphisms:

$$H^1_{fppf}(\mathcal{C},\mathcal{Q},\mathcal{G}) \simeq \mathrm{Hom}(\mathcal{G}^D, \mathrm{Pic}_{\mathcal{C}/R}) \simeq \mathrm{Hom}(\mathcal{G}^D, \mathrm{Pic}^0_{\mathcal{C}/R}).$$

This reduces the question of extending torsors into extending group functors and morphisms between them.

Theorem (M. [3])

We have an isomorphism:

$$H^1_{klf}(\mathcal{C},\mathcal{Q},\mathcal{G}) \simeq \mathrm{Hom}(\mathcal{G}^D, \mathrm{Pic}^{log}_{\mathcal{C}/R}).$$

Question: Can we replace $\mathrm{Pic}^{log}_{\mathcal{C}/R}$ by an object which is easier to handle in practice?

$\mathrm{Pic}^{log}_{\mathcal{C}/R}$ has the Néron property $\implies$ $J = \mathrm{Pic}^0_{C/K} \hookrightarrow \mathrm{Pic}_{C/K}$ extends uniquely to $\mathcal{J} \to \mathrm{Pic}^{log}_{\mathcal{C}/R}$, where $\mathcal{J}$ is the Néron model of J.

Corollary (M. [3])

Let $Y \to C$ be a pointed fppf G-torsor. If there exists a finite flat R-group scheme $\mathcal{G}$ with generic fiber G and such that the associated K-morphism $G^D \to J$ (cf.$(*)$) extends into an R-morphism $\mathcal{G}^D \to \mathcal{J}$, the torsor extends uniquely into a pointed log $\mathcal{G}$-torsor over any regular model of C. In addition, the extended log torsor is fppf if and only if $\mathcal{G}^D \to \mathcal{J}$ factors through $\mathcal{J}^0$.

- If C is semistable, one can replace $\mathrm{Pic}^{log}_{\mathcal{C}/R}$ by $\mathcal{J}$ in the above theorem and the previous corollary becomes a criterion of extension.

Question: Can one find examples of finite commutative K-group schemes for which one can apply the previous corollary ?

For any integer $r > 1$, if $\mathcal{J}[r]$ is a finite flat R-group scheme, then C is semistable: let Γ denote its dual graph. Conversely, if the number of edges common to any two circuits of Γ is always a multiple of r, $\mathcal{J}[r]$ is finite and flat (cf. [1] and [3]).

Theorem (M. [3])

Assume that C is semistable with dual graph Γ, that G is killed by r and that the number of edges common to any two circuits of Γ is always a multiple of r. Let $Y \to C$ be a pointed fppf G-torsor such that Y is geometrically connected.Then the schematic closure $\mathcal{F}$ of G^D in $\mathcal{J}[r]$ is finite and flat and $Y \to C$ extends uniquely into a pointed log $\mathcal{F}^D$-torsor over any regular model of C.

Question: Can we still say something if G does not admit a finite flat R-model ?
If R is assumed to be Henselian Japanese with perfect residue field, any quasi-finite flat torsor reduces to a finite flat torsor ([2]).

Theorem (M. [4])

If R is assumed to be Henselian Japanese with perfect residue field and if $Y \to C$ is an fppf pointed G-torsor, there exists a quasi-finite flat R-group scheme $\mathcal{G}$ with generic fiber G, and an fppf pointed $\mathcal{G}$-torsor extending Y over any regular model of C.

Example of application

Consider the smooth projective curve C over $\mathbb{Q}$ covered by the two affine charts:

- $y^2 = f(x) = x^6 - 10x^3 + 9$,
- $t^2 = s^6 f(\frac{1}{s})$,

that are glued together using $(x,y) = (\frac{1}{s}, \frac{t}{s^3})$.
Pointed fppf μ_3^2-torsor over C: Consider the two divisors:

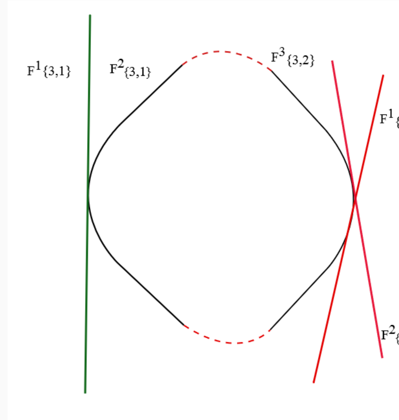
$$\mathrm{div}(y - x^3 - 3) = 3(0,3) - 3\infty,$$
$$\mathrm{div}(y - x^3 + 3) = 3(0,-3) - 3\infty,$$

where ∞ denotes one of the two points at infinity on C.
The two divisors

$$D_+ = (0,3) - \infty \quad \text{and} \quad D_- = (0,-3) - \infty$$

define two classes of order 3 in the Jacobian of C that are independent classes of divisors $\implies$ The Jacobian of the curve C contains a subgroup isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2 \overset{(*)}{\implies}$ We have a pointed fppf μ_3^2-torsor over C (relatively to $(1,0)$).

Construction of a regular model of C over $\mathbb{Z}_3$: Using the Jacobian criterion, we check that in each affine chart, there are two singular points in the fiber over the prime 3, which in fact belong to the intersection of the two charts. So we blow up once at those two singular points and fortunately obtain a regular model $\mathcal{C}_3$ of C over $\mathbb{Z}_3$.



The special fiber of $\mathcal{C}_3$

Study of the extension of the pointed fppf μ_3^2-torsor:

- $(\mathbb{Z}/3\mathbb{Z})^2 \to J$ extends by Néron property to $(\mathbb{Z}/3\mathbb{Z})^2 \to \mathcal{J} \implies$ The μ_3^2-torsor over C extends into a log μ_3^2-torsor over $\mathcal{C}_3 \implies D_+$ and D_- extend into log $\mathbb{G}_m$-torsors over $\mathcal{C}_3$ ($\mathbb{Q}$-divisors):

$$\overline{D}_+ = \frac{1}{3}\mathrm{div}(y - x^3 - 3) = \overline{(0,3)} - \overline{\infty} + \frac{1}{3}V_3^+,$$
$$\overline{D}_- = \frac{1}{3}\mathrm{div}(y - x^3 + 3) = \overline{(0,-3)} - \overline{\infty} + \frac{1}{3}V_3^-.$$

- The extended log torsor lifts into an fppf torsor $\iff$ $(\mathbb{Z}/3\mathbb{Z})^2 \to \mathcal{J}$ factors through $\mathcal{J}^0 \simeq \mathrm{Pic}^0_{\mathcal{C}_3/R} \iff \overline{D}_+$ and $\overline{D}_-$ have integer coefficients.
- After computing $\mathrm{div}(y - x^3 - 3)$ and $\mathrm{div}(y - x^3 + 3)$ on the special fiber of $\mathcal{C}_3$, we find out that:

$$\overline{D_+} = \overline{(0,3)} - \overline{\infty} + \frac{1}{3}(2F^1_{3,2} + F^2_{3,2}) \quad \text{and} \quad \overline{D_-} = \overline{(0,-3)} - \overline{\infty} + \frac{1}{3}(2F^2_{3,2} + F^1_{3,2})$$

Conclusion: The pointed fppf μ_3^2-torsor over C extends into a pointed log μ_3^2-torsor over $\mathcal{C}_3$ which is not fppf.

References

[1] A. Chiodo: *Quantitative Néron theory for torsion bundles*; Manuscripta Mathematica (2009).
[2] P. Hô Hai, P.J Dos Santos: *Finite torsors on projective schemes defined over a discrete valutaion ring*; Algebraic Geometry 10 (1) (2023).
[3] S. Mehidi: *Extending torsors via regular models of curves*; Manuscripta Mathematica (2022).
[4] S. Mehidi: *Extending torsors under quasi-finite flat group schemes*; https://arxiv.org/abs/2211.16067v2.