I_ Category theory

- Category : A category C is the data of
  - obj : collection of objects
  - arrows : For every two objects $a, b \in C$, a collection of arrows $f : a \to b$ (Can be empty)
  - For every $a \in Obj(C)$, $a \to a$ identity morphism.
  - For every, $a, b, c \in obj(C)$, $f : a \to b$ and $g : b \to c$, an arrow "composite" $g \circ f : a \to c$.

+ conditions of unitality and associativity

$$\begin{tikzcd} a \arrow[r, "f"] & b \\ a \arrow[u, "id"] \end{tikzcd} = a \to b \qquad \begin{tikzcd} a \arrow[r, "f"] & b \arrow[d, "id"] \\ & b \end{tikzcd} = a \to b$$

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

$$h \circ (g \circ f) = (h \circ g) \circ f := h \circ g \circ f$$

Eg : • (Set) : composition of function
• (Gp)

- X a topological space. (Open of X)

obj : open sets — arrows : $U \subseteq V$

(For two open sets, we have an arrow $U \to V$ iff $U \subseteq V$)

- **Functors:**

$\mathcal{C}$, $\mathcal{D}$ two categories.

$F : \mathcal{C} \to \mathcal{D}$ a functor is a pair of assignments

$$ob(\mathcal{C}) \xrightarrow{F} ob(\mathcal{D})$$

$$Arrow(\mathcal{C}) \xrightarrow[a \to b]{F} Arrow(\mathcal{D}) \qquad F(a) \to F(b)$$

Functoriality axioms

st $\quad a \xrightarrow{id} a \longmapsto F(a) \xrightarrow{F(id)} F(a)$ is the identity on $F(a)$

$$a \xrightarrow{f} b \xrightarrow{g} c \longmapsto F(a) \xrightarrow{F(f)} F(b) \xrightarrow{F(g)} F(c)$$

$$a \xrightarrow{g\circ f} c \longmapsto F(a) \xrightarrow{F(g\circ f)} F(c)$$

$$F(g) \circ F(f) = F(g \circ f)$$

Eg: $\cdot$ $(Gp) \to (Set)$ forgetful functor

$\cdot$ $(\text{Top spaces}) \to (Gp)$ $\quad X \longmapsto \pi_1(X)$.
the fundamental gp.

Definition: (Full / faithfull functor)

$F : \mathcal{C} \to \mathcal{D}$ a functor.
let $a, b \in \mathcal{C}$

$$Hom(a,b) \longrightarrow Hom(F(a), F(b))$$
$$f \longmapsto F(f)$$

- Full if this surjective $\quad \forall\, a,b \in ob(\mathcal{C})$.
- Faithful if this is injective $\quad$ //.
- Fully-faithful if surj + inj $\quad$ //.

Eg: The forgetful functor is faithful. But not full.

$$\begin{tikzcd} G \arrow[r, "f"] & H \end{tikzcd} \qquad\qquad \underline{G} \longrightarrow \underline{H}$$

defined on sets + respects gp structure

Question? $Top \to (Gp)$

$X \longmapsto \pi(X)$

Definition (opposite category): "flipping all arrows"

check this is a category

Let C be a category.

We can construct from C a category C^{op}: the opposite category

- $ob(C^{op}) = ob(C)$
- arrows: if $f: a \to b$ an arrow in C

then $b \to a$ is an arrow in C^{op}

Definition (Covariant / Contravariant functor)

Let C be a category.

A contravariant functor from C to D is a functor from C^{op} to D:

$$C^{op} \xrightarrow{F} D$$

$$a \xrightarrow{f} b \longmapsto F(b) \xrightarrow{F(f)} F(a)$$

arrow in C

Eg:
Fix a gp G.

$$\mathrm{Hom}: (Gp)^{op} \to (Set)$$
$$H \mapsto \mathrm{Hom}(H, G)$$
$$H \to H' \mapsto \mathrm{Hom}(H', G) \to \mathrm{Hom}(H, G)$$
$$\mathrm{Hom}(H') \to \mathrm{Hom}(H)$$

is contravariant functor.

- Natural transformations

$F, G : C \to D$ functors

the collection of arrows $\{F(c) \xrightarrow{\eta_c} G(c) \mid c \in Ob(C)\}$

is called a natural transformation if for each arrow

$f: a \to b$ of object in C, we have a com. diag:

$$\begin{tikzcd} F(a) \arrow[r, "\eta_a"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\circlearrowright"] & G(a) \arrow[d, "G(f)"] \\ F(b) \arrow[r, "\eta_b"'] & G(b) \end{tikzcd}$$

if for every $a \in Ob(C)$, η_a is an isomorphism in D, F and G are isomorphic: $F \simeq G$.

Motivation

Let X be a scheme

We get a functor $h_X : (Sch)^{op} \to (Set)$ functor of pt's

$$Y \mapsto \mathrm{Hom}(Y, X)$$

the essence of an object in a category is determined by how it is related to the other object.
What you need to remember about an object X is contained in h_X (this encode the morphisms from Y to X).

Lemma: (Yoneda lemma)

The functor $h_{-} : (Sch) \to Fun(Sch^{op}, Set)$

$$X \mapsto h_X$$

is full faithfull, i.e

$$Hom(X, Y) \cong Hom(h_X, h_Y)$$

you don't loose any info by looking at map of functor of pt

This is not an equivalence of categories! We call the elements in the image "representable functors".

↳ motivate the following def.

Definition: (Representable functor)

A functor $F : Sch^{op} \to Set$ is said to be representable if $F \simeq h_X$ for some $X \in Sch$.

When you represent a functor F, you give a scheme X + an isomorphism $F \simeq h_X$: X is unique up to unique isomorphism by the Yoneda lemma.

(X, ϕ_X) $\quad$ (Y, ϕ_Y)

$F \xrightarrow[\sim]{\phi_X} h_X$ $\quad$ $F \xrightarrow[\sim]{\phi_Y} h_Y$

$h_X \xrightarrow{\phi_Y \circ \phi_X^{-1}} h_Y$ ⇝ Yoneda lemma $\exists !$ isomorphism $X \simeq Y$.

(Functor preserves isomorphisms)

Example 1:

$$\mathbb{A}^n : (Sch)^{op} \longrightarrow (Set)$$
$$Y \longmapsto \Gamma(Y, \mathcal{O}_Y)^n$$
$$Y \to Y' \longmapsto \Gamma(Y', \mathcal{O}_{Y'})^n \to \Gamma(Y, \mathcal{O}_Y)^n$$

Then $X = \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]$ represents this functor:

$\mathbb{A}^n \simeq h_X$?

$$h_X(Y) = \operatorname{Hom}(Y, X) \cong \operatorname{Hom}(\mathbb{Z}[x_1, \dots, x_n], \Gamma(Y, \mathcal{O}_Y))$$
$$\cong \Gamma(Y, \mathcal{O}_Y)^n$$
$$= \mathbb{A}^n(Y)$$

Example 2:

$$\mathcal{M}_{1,1} : (Sch)^{op} \longrightarrow (Set)$$
$$Y \longmapsto \{\text{iso. classes of elliptic curves on } Y\}$$

$$Y' \to Y \longmapsto \mathcal{M}_{1,1}(Y) \to \mathcal{M}_{1,1}(Y')$$
$$E \xrightarrow{g} Y \longmapsto g^*E \to Y'$$

is an elliptic curve:

- fibers are genus 1 curves

- Section:

$$\begin{tikzcd} Y' \arrow[r] & Y \\ g^*E \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & E \arrow[u, "g"'] \\ Y' \arrow[u, dashed] \arrow[r] \arrow[uu, bend left, "="] & Y \arrow[u, "e"'] \end{tikzcd}$$

$e \circ g = id_Y$

Proposition: $\mathcal{M}_{1,1}$ is not representable

Lemma: (Next session!)

A representable functor is an fppf sheaf.

Proof:

Idea: Consider elliptic curves over $\mathbb{Q}$: Seen as schemes over $\operatorname{Spec}\mathbb{Q}$.
Take a cover U of $\operatorname{Spec}(\mathbb{Q})$ (eg étale cover)
Take $s, s' \in \mathcal{M}_{1,1}(\operatorname{Spec}\mathbb{Q})$: choice of a class of elliptic curve
st $s|_U = s'|_U$:
if $\mathcal{M}_{1,1}$ is a sheaf, we should have $s = s'$

Counter-example:

W. model — make change of var to get W. model

E_1: $y^2 = x^3 + x + 1$ $\qquad$ E_2: $3y^2 = x^3 + x + 1$

isomorphic if W models are

are not isomorphic over $\mathbb{Q}$.

But are isomorphic over $\mathbb{Q}(\sqrt{3})$.

$\operatorname{Spec}\mathbb{Q}(\sqrt{3}) \longrightarrow \operatorname{Spec}\mathbb{Q}$ surjective + étale : cover in the "étale" topology

Let $s \in \mathcal{M}_{1,1}(\operatorname{Spec}\mathbb{Q})$ be the section defined by E_1

$s' \in \mathcal{M}_{1,1}(\operatorname{Spec}\mathbb{Q})$ " " " E_2

$$s \in \mathcal{M}_{1,1}(\mathbb{Q}) \longrightarrow s|_{\mathbb{Q}(\sqrt{3})} \in \mathcal{M}_{1,1}(\mathbb{Q}(\sqrt{3}))$$
$$\| $$
$$s'|_{\mathbb{Q}(\sqrt{3})}$$

But $s \neq s'$

We say that $M_{1,1}$ doesn't have a fine moduli space.

More generally a functor $M: (Sch)^{op} \to (Set)$ occuring in moduli theory, can in general be lifted to a functor : $\mathcal{M} : (Sch)^{op} \to (Groupoids)$

If $M = M_{1,1}$, $\mathcal{M}$ is obtained by sending Y to the groupoid whose objects are the iso classes of elliptic cures $/Y$ and whose morphisms are isomorphisms between them.

$\mathcal{M}$ is called a stack. In most situations, even if M is not fine moduli space, $\mathcal{M}$ will be "nice" enough to work with.

IV - Grothendieck topologies :

- A Grothendieck topology is a structure on a category C that makes the objects on C act as open sets of a topological space.
- X a top space. Instead of focusing on the opens themselves as covers, we look at the morphisms from the open to X, i.e. the inclusion.
- This allowed him to axiomatize the notion of covers
- We define a cover by considering morphisms $U_i \to X$ which are "nice" enough to give a good notion of a cover.

Definition: (Grothendieck topology)

Let C be a category

1) A Grothendieck topology on C consists of, for each object X of C, a collection $Cov(X)$ of sets $\{X_i \to X\}$ of arrows, called coverings of X st

   1. If $V \to X$ is an isomorphism, $\{V \to X\} \in Cov(X)$.
   2. If $\{X_i \to X\} \in Cov(X)$ and $Y \to X$ any arrow, $\{X_i \times_X Y \to Y\} \in Cov(Y)$.
   3. If $\{X_i \to X\} \in Cov(X)$, $\{V_{ij} \to X_i\} \in Cov(X_i)$, $\{V_{ij} \to X\} \in Cov(X)$

2) A site is a category + Grothendieck topology: (C, Cov).

Etale: is an algebraic analogue of the notion of a local isomorphism in the complex analytic topology.

Eg: A Galois cover is étale.

Examples:

1) Small Classical topology:

X a topological space. $Op(X)$ the category of open sets in X, and $\mathrm{Hom}(U,V) = \begin{cases} U \hookrightarrow V \text{ if } U \subseteq V \\ \emptyset \text{ else} \end{cases}$

For $U \in Op(X)$, we define $Cov(U)$ to be the collection $\{U_i \to U\}_{i \in I}$ for which $U = \bigcup_i U_i$. Cover in the usual sense

In particular, if X is a scheme, the Zariski topology defines the "small Zariski site".

2) Big Classical topology

C a category of topological spaces, with morphisms continuous maps.

For X a topological space, define $Cov(X)$ to be the collections of families $\{X_i \to X\}_{i \in I}$ st $X_i \to X$ is open and $\bigcup X_i = X$.

↳ Big Zariski site:

Let X be a scheme, $C = (Sch/X)$

For $(U \to X)$ in C, define $Cov(U)$ to be the set of collections of X-morphisms $\{U_i \to U\}_{i \in I}$ st $U_i \hookrightarrow U$ and $U = \bigcup_i U_i$.

the difference is that in the big site, morphisms are given by cont. maps and small site by "cano" fppf

3) <u>Small étale site:</u>

X a scheme.

Let $Et(X)$ be the full subcategory of the category of (Sch/X) whose objects are étale maps $U \to X$.

A collection of maps $\{U_i \to U\}_{i \in I}$ is in $Cov(U)$ if the map

$$\coprod_{i \in I} U_i \to U$$

is surjective.

(Rmk: all morphisms in $Et(X)$ are étale)

fppf

4) <u>Big étale site:</u>

X a scheme.

Consider (Sch/X).

A collection of maps $\{U_i \to U\}_{i \in I}$ is in $Cov(U)$ if each $U_i \to U$ is étale and

$$\coprod_i U_i \to U$$

is surjective.

5)

Let Δ be a category and C a site. Let $F: \Delta^{op} \to C$ be a functor.

Define C_F as follows:

<u>Ob</u> (C_F): pairs $(\delta, X \to F(\delta))$ where $\delta \in \Delta$.

map in C

<u>arrows</u> : $(\delta', X' \to F(\delta')) \to (\delta, X \to F(\delta))$

is a pair $\begin{cases} f : \delta \to \delta' & \text{a morphism in } \Delta. \\ f^b : X \to X' & \text{" " " } C \end{cases}$

st

$$\begin{tikzcd} X' \arrow[r, "f^b"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d] \\ F(\delta') \arrow[r, "F(f)"'] & F(\delta) \end{tikzcd}$$

• Define coverings in C_F :

For $(\delta, X \to F(\delta)) \in Ob(C_F)$, the set of coverings is given by the set of collections

$$\{(\delta, X_i) \xrightarrow{id,\, f_i^b} (\delta, X)\}_{i \in I}$$

st $\{X_i \to X\} \in Cov(X)$ in C.

Eg : Take $\Delta = \{*\}$ and

$$F : \{*\} \to C$$
$$* \longmapsto X_0 \text{ some object in } C$$
$$* \xrightarrow{id} * \longmapsto X_0 \xrightarrow{id} X_0$$

C_F : $\begin{cases} \underline{ob} : Y \to X_0 & \text{(obj over } X_0\text{)} \\ \underline{arrows} : Y' \to X_0 \longrightarrow Y \to X_0 \end{cases}$ (*)

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] \arrow[dr, phantom, "\circlearrowright"] & Y' \arrow[d] \\ & X_0 \end{tikzcd}$$

coverings: if $Y \to X_0$ object in C_F

$$\left\{ \begin{array}{c} Y_i \to Y \\ \searrow X_0 \swarrow \end{array} \;\middle|\; Y_i \to Y \in \mathrm{Cov}(Y) \text{ in } C \right\}$$

Take (C, Cov) a site and $X_0 \in \mathrm{ob}(C)$

$\leadsto$ induces a site on $(C/_{X_0})$ (*)

IV - Sheaves and limits

Definition: (Sheaf)

Let C be a category.

A presheaf of sets on C is a functor $F : C^{op} \to \mathrm{Set}$.

If C is a site, then F is a sheaf on (C, Cov) if it is a presheaf st for every object X in C and every covering $\{X_i \to X\} \in \mathrm{Cov}(X)$, the sequence

$$F(X) \to \prod_i F(X_i) \overset{pr_1^*}{\underset{pr_2^*}{\rightrightarrows}} \prod_{i,j} F(X_i \times_X X_j)$$

is exact.

$\forall i \quad X_i \to X \leadsto F(X) \to F(X_i)$

$X_i \times_X X_j \xrightarrow{pr_1} X_i$

$F(X_i) \xrightarrow{pr_1^*} F(X_i \times X_j)$

$F(X_j) \xrightarrow{pr_2^*}$

$s_i \in F(X_i)$ for each i st $s_i|_{X_i \times X_j} = s_j|_{X_i \times X_j}$

$s_i|_{X_i \times X_j} \quad F(X_i \times X_j)$

then $\exists!\, s \in F(X)$ st $s|_{X_i} = s_i$.

Rmk : if X is a topological space and C the small site on X, this is the usual sheaf condition

Rmk : (Sheaves) $\to$ (presheaves)

$$F \longmapsto \underline{F}$$

forgetful functor is fully faithful.

This is not an equivalence of categories!

Counter-example :

$X = \{p, q\}$ + discrete topology, S a set st $|S| > 1$

Let $F : Op(X)^{op} \to Set$ st $F(U) = S$.

This is not a sheaf!

Take $s \neq s' \in S = F(\emptyset)$

They agree in the restrictions (empty condition) $\Rightarrow s = s'$

Contradiction. But one can "sheaffify" a presheaf in a canonical way.

adjunction : weaker than eq of cat

$D \xrightarrow{F} C$, $C \xrightarrow{G} D$ if $\underbrace{Hom(FY, X)}_{\in C} = \underbrace{Hom(Y, GX)}_{\in D}$

Theorem : (Sheaffification)

The forgetful functor from sheaves to presheaves has a left adjoint. In particular, if F is a presheaf, there is a morphism to a sheaf $F \to F^a$ so that any morphism to a sheaf factors via F^a. (F left adj to G $\rightsquigarrow$ $Hom(FX, Y) = Hom(X, GY)$)

Proof:

$*$ (Separated presheaves on C) $\overset{ff}{\hookrightarrow}$ (presheaves on C)

$**$ (sheaves on C) $\overset{ff}{\hookrightarrow}$ (separated presheaves on C)

we show that $*$ and $**$ have both left adjoints, then their composition gives the desired functor $F \mapsto F^a$.

$(*)$ Let F be a presheaf on C, define F^s to be the quotient of F, which sends

$$U \longmapsto F(U)/\sim$$

where $a \in F(U) \sim b \in F(U)$ if $\exists \{U_i \to U\} \in Cov(U)$

st $a \in F(U) \mapsto a_i \in F(U_i)$

$$\begin{tikzcd}[row sep=small] a\in F(U) \arrow[r, mapsto] & a_i \arrow[d, phantom, "\parallel"] \in F(U_i) \\ b \in F(U) \arrow[r, mapsto] & b_i' \in F(U_i) \end{tikzcd}$$

Question: Why is F^s a presheaf?

$$V \xrightarrow{f} U \overset{?}{\rightsquigarrow} F^s(U) \to F^s(V)$$

F is functor

$$\begin{tikzcd} F(U) \arrow[r, "F(f)"'] \arrow[d] & F(V) \arrow[r] & F(V)/\sim \\ F(U)/\sim \arrow[urr, dashed, "?"'] & & \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[d, hook] & V \times_U U_i \arrow[l] \arrow[d] \\ U & U_i \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} F(V) \arrow[r] & F(V \times_U U_i) \\ F(U) \arrow[u] \arrow[r] & F(U_i) \arrow[u] \end{tikzcd}$$

take $a, b \in F(U)$ st $a_i = b_i$ $\quad \in Cov(V)$

We have a comm. diag

$$\begin{tikzcd} F(f)(a),\ F(f)(b) \quad F(V) \arrow[r] & \prod_i F(V \times_U U_i) \quad a_i|_V = b_i|_V \\ F(U) \arrow[u, "F(f)"] \arrow[r] & \prod F(U_i) \arrow[u] \end{tikzcd}$$

$a \quad b \qquad\qquad a_i = b_i$

So F^s is a presheaf.

In addition, by cst if G is separated presheaf, then any map $F \to G$ factors via F^s.

(**) Let F be a separated sheaf on C.

We define F^a to be the presheaf st

$$U \mapsto F^a(U) := \left(\{U_i \to U\}_{i\in I}, \{a_i\} \right) \Big| \, a_i \in \mathrm{Eq}\left(\prod F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times U_j) \right) \Big/ \sim$$

$$\left(\{U_i \to U\}_{i\in I}, \{a_i\} \right) \sim \left(\{V_j \to U\}_{j\in J}, \{b_j\} \right)$$

if a_i and b_j have same image in $F(U_i \times V_j)$ for all $i \in I$, $j \in J$.

<u>Question</u>: – Show it is a presheaf / sheaf + left adjoint.

(*) Limits:

Let C be a category, $F: I \to C$ a functor.

For $X \in C$, define $k_X : I \to C$ to be the functor sending each object to X and each morphism to id_X.

We define $\varprojlim F : C \to \mathrm{Set}$ as the functor given by $X \mapsto \mathrm{Nat}(k_X, F)$.

Question: is this functor representable ?

Examples :

1) Consider the category I of two objects A and B and two maps $A \rightrightarrows B$.

$$F : I \to C \qquad\qquad k_X : I \to C$$
$$A \mapsto F(A) = X_1 \qquad\qquad A \mapsto X$$
$$B \mapsto F(B) = X_2 \qquad\qquad B \mapsto X$$
$$A \rightrightarrows B \mapsto X_1 \underset{f_2}{\overset{f_1}{\rightrightarrows}} X_2 \qquad\qquad A \rightrightarrows B \mapsto X \underset{id}{\overset{id}{\rightrightarrows}} X$$

$\varprojlim F : C \to \mathrm{Set}$

$X \mapsto \mathrm{Nat}(k_X, F)$

$$\begin{tikzcd} X = k_X(A) \arrow[r, "g"] \arrow[d, "id"'] \arrow[dr, phantom, "\circlearrowright"] & F(A) = X_1 \arrow[d, "f_1"] \\ X = k_X(B) \arrow[r, "h"'] & F(B) = X_2 \end{tikzcd}$$

$$\begin{tikzcd} k_X(A) = X \arrow[r, "g"] \arrow[d, "id"'] \arrow[dr, phantom, "\circlearrowright"] & X_1 = F(X) \arrow[d, "f_2"] \\ k_X(B) = X \arrow[r, "h"'] & X_2 = F(B) \end{tikzcd}$$

$$f_2 \circ g = f_1 \circ g$$

$\varprojlim F(X) = \{ X \xrightarrow{g} X_1 \mid f_1 \circ g = f_2 \circ g \}$

$\varprojlim F$ is represented by the equalizer of f_1 and f_2.

$\varprojlim F(X) = h_{eq}(X) = \mathrm{Hom}(X, eq)$

$eq(f_1, f_2) = \{ Y \subset X_1 \mid f_1|_Y = f_2|_Y \}$

$= \{ X \to Y \}$

$= \{ X \xrightarrow{g} X_1 \mid f_1 \circ g = f_2 \circ g \}.$

2) I a set and $F(I) = \{X_i\}_{i \in I}$

then $\varprojlim F(X) = \{ \{g_i : X \to X_i\}_{i \in I} \}$

this functor is represented by the product $\prod_{i \in I} X_i$

3) I the category of three objects and two maps

$\bullet \longrightarrow \bullet \longleftarrow$

$F(I) = X_1 \xrightarrow{f_1} X_2 \xleftarrow{f_3} X_3$

then $\varprojlim F(X) = \{ X_1 \xleftarrow{g_1} X \xrightarrow{g_3} X_3 \mid f_1 \circ g_1 = f_3 \circ g_3 \}$

It's represented by the fiber product $X_1 \times_{X_2} X_3$

Lemma:

C a category. TFAE

(1) Projective limits in C are representable ($\varprojlim F$).

(2) Products and equilizers are representable

(3) Products and fiber products are representable.

$1 \Rightarrow 2$ ok.

$\Rightarrow 3$

(*) Definition (topos)

A topos is a category that "behaves" like the category of sheaves on a top. space

A topos is a category equivalent to the category of sheaves on a site.

Rmk: Different sites may induce equivalent topos.

eg: $Et^{aff}(X)$ Subcategory of $Et(X)$ consisting of étale morphisms $U \to X$ with U affine.

$\{U_i \to U\}_{i \in I}$ is a covering in $Et^{aff}(X)$ if it is a covering in $Et(X)$.

Then $Et^{aff}(X)$ is a site whose induced topos is equivalent to the topos of sheaves on $Et(X)$.

"Sheaves on $Et(X)$" and "Sheaves on $Et^{aff}(X)$" are the same.

- Morphism of topos:

$f: T \to T'$ is an isom. class of triples (f_*, f^*, ϕ) where $f_*: T \to T'$, $f^*: T' \to T$ are functors, and ϕ an adjunction between them:

$$\phi: \mathrm{Hom}_T(f^* a, b) \xrightarrow{\sim} \mathrm{Hom}_{T'}(a, f_* b)$$

+ f^* commute with limits.

Eg: $X \xrightarrow{f} X'$ continuous map of top spaces.

- $f^{-1}: \mathrm{Open}(X') \to \mathrm{Open}(X)$
- Define the functor: Open (X) is a site

$$f_*: (Sh/X) \to (Sh/X')$$
$$F \longmapsto (f_* F)(U) := F(f^{-1}(U))$$

this has a left adjoint f^* which commutes with proj limits

Open (X) is a site (cat + Covers)
↳ define sheaves on X
Topos: (Sh/X) ⇝ f_* is a map of topos

Proposition: T a topos and $F: I \to T$ a functor. Then $\varprojlim F$ is representable

↳ Proof: (⇐) prove that products and eq are representable.
By definition, T is equivalent to the category of sheaves on a site C.
Let $\{F_i\}_{i \in I}$ a set of sheaves on C.
Their product $\prod F_i$ is represented by the sheaf (on C):

$$U \in \mathrm{obj}(C) \mapsto \prod F_i(U)$$

(show this is a sheaf) ∎