S scheme

Recall: a stack $\mathcal{X}$ is algebraic if

(i) $\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times_S \mathcal{X}$ is representable

(ii) there exists a smooth surjective morphism $\pi : X \to \mathcal{X}$ where X is a scheme

Def An algebraic stack is Deligne-Mumford (DM) if π in (ii) can be taken to be étale.

This can be also be restated in terms the diagonal, using the following notion

Def A morphism of schemes $g : Z \to W$ is formally unramified if for every closed embedding $S_0 \hookrightarrow S$ defined by a nilpotent ideal the map

$$Z(S) \longrightarrow Z(S_0) \times_{W(S_0)} W(S)$$

is injective

Equivalent to $\Omega^1_{Z/W} = 0$.

Therefore, this property is stable with respect to the smooth topology and local in domain for the étale topology

Thus makes sense to consider for representable morphisms.

Theorem $\mathcal{X}/S$ algebraic stack.
Then $\mathcal{X}$ is DM if and only if
$\Delta: \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is formally unramified
(skip proof)

Can interpret this as the statement that $\mathcal{X}$ is DM if and only if the objects of $\mathcal{X}$ do not admit infinitesimal automorphisms.

Corollary $\mathcal{X}/S$ algebraic stack.
If $\Delta: \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is of finite presentation,
(automatic if $\mathcal{X}/S$ of finite presentation)
then: Δ is formally unramified
$\iff$ for every algebraically closed field k & $x \in \mathcal{X}(k)$, $\underline{Aut}_x$ is a reduced finite k-group scheme.

Proof: $\Rightarrow$

$\underline{\mathrm{Aut}}_x$ is formally unramified over k as it is the fiber product of

$$\begin{tikzcd} & \operatorname{Spec} k \arrow[d, "(x,x)"] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

Therefore $\underline{\mathrm{Aut}}_x$ is a disjoint finite union of copies of $\operatorname{Spec} k$

$\Leftarrow$: For any morphism $U \to \mathcal{X} \times_S \mathcal{X}$ consider fiber product

$$\begin{tikzcd} P \arrow[r] \arrow[d] & U \arrow[d] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

$P \to U$ is formally unramified iff for every $z \in U$, the fiber $P_z \to \operatorname{Spec}(k(z))$ is formally unramified, this is true if and only if the base change $P_\Omega \to \operatorname{Spec}(\Omega)$ is formally unramified, where Ω is an algebraic closure.

Now P_Ω is empty or $P_\Omega \cong \underline{\mathrm{Aut}}_x$ where $x = \mathcal{X}(\Omega)$ is the pullback of u_1:

$$\begin{tikzcd} U \arrow[rr] \arrow[dr, "\delta"'] & G & \mathcal{X} \\ & \mathcal{X} \times_S \mathcal{X} \arrow[ur, "\mu_1"'] & \end{tikzcd}$$

Therefore if each of the $\underline{\mathrm{Aut}}_x$ are formally unramified, then the diagonal is as well. $\square$

<u>Corollary</u> Let $\mathcal{X}/S$ be an algebraic stack such that for every S-scheme T and every object $x \in \mathcal{X}(T)$, the automorphism group of x is trivial. Then $\mathcal{X}$ is an algebraic space.

<u>Proof</u> Under this assumption $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by monomorphisms, so $\mathcal{X}$ is DM. Therefore we have an étale surjection $U \to \mathcal{X}$ with U a scheme, and $R = U \times_{\mathcal{X}} U \to U \times_S U$ is an étale equivalence relation on U such that

$\mathcal{X} = U/R$. $\square$

Remark: With more work, one can show that we only need to consider geometric points on $\mathcal{X}$ (so morphisms $\mathrm{Spec}\, k \to \mathcal{X}$ with k algebraically closed.)

Examples

X scheme $/S$

G/S smooth group scheme acting on X

we consider the algebraic stack $[X/G]$

Can describe when $[X/G]$ is DM in terms of the action of G.

Let $\bar{s}: \mathrm{Spec}(k) \to S$, k algebraically closed

$G_{\bar{s}}$ pullback of G along $\bar{s}$.

Consider $(P, \pi) \in [X/G](k)$

$G_{\bar{s}}$-torsor $\pi: P \to X_{\bar{s}}$ equivariant

Since k is algebraically closed, P is a trivial torsor.

Fix a trivialisation $p \in P(k)$ and write $x = \pi(p) \in X(k)$

We get a map

$$\lambda : G_{\bar{s}} \to X_{\bar{s}}, \quad g \mapsto g \cdot x$$

The fiber product G_x of the diagram

$$\begin{tikzcd} & \operatorname{Spec} k \arrow[d, "x"] \\ G_{\bar{s}} \arrow[r, "\lambda"] & X_{\bar{s}} \end{tikzcd}$$

is isomorphic to $\underline{\mathrm{Aut}}_{(P,\pi)}$

For different trivialisations of P, the resulting subgroup schemes of $G_{\bar{s}}$ are conjugate.

Thus for a point $t \in [X/G](k)$ it makes sense to talk about the stabiliser group scheme $G_t \subset G_{\bar{s}}$

<u>Corollary</u> The stack $[X/G]$ is DM if and only if for every point $\bar{s} : \operatorname{Spec}(k) \to S$ as above and $t \in [X/G](k)$, the stabiliser group scheme $G_t \subset G_{\bar{s}}$ is étale over $\bar{s}$.

<u>Proof</u> For a field L and a morphism $z : \operatorname{Spec} L \to [X/G]$ with automorphism group scheme $\underline{\mathrm{Aut}}_z$, $\underline{\mathrm{Aut}}_z / L$ is unramified if and only if $\underline{\mathrm{Aut}}_z \times \bar{L} / \bar{L}$ is étale.

Therefore $[X/G]$ is DM if all stabilisers G_t are étale. $\square$

Moduli of curves

Fix $g \geq 2$ recall the fibered category M_g over Sch whose objects are pairs $(S, f: C \to S)$, where S is a scheme and f is a proper smooth morphism such that every geometric fiber is a connected genus g curve.

A morphism

$$(S', f': C' \to S') \longrightarrow (S, f: C \to S)$$

in M_g is a Cartesian square

$$\begin{tikzcd} C' \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{f}"'] & C \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

$$\begin{aligned} M_g &\longrightarrow Sch \\ (S, f) &\longmapsto S \end{aligned}$$

makes M_g a fibered category.

Theorem $\mathcal{M}_g$ is a separated smooth DM stack over $\operatorname{Spec} \mathbb{Z}$.

Skip proof (DM is in Olsson, full statement is in J. Alpers draft on stacks.)

Remark If $g=0,1$ we can still define $\mathcal{M}_g$, but for $g=1$ there is a subtlety: there exist a family of genus 1 curves $C \to S$ such that C is not a scheme but an algebraic space. By allowing this $\mathcal{M}_g$ is a smooth algebraic stack over $\operatorname{Spec} \mathbb{Z}$, but it is not DM for $g=0,1$.

$\mathcal{M}_0 \cong BPGL_2$

Root stacks

Given an effective Cartier divisor D on a scheme X and a positive integer n, want to construct a stack $(X, \sqrt[n]{D})$ such that a morphism
$$S \longrightarrow (X, \sqrt[n]{D})$$ roughly corresponds with a morphism $f : S \to X$ and an effective Cartier divisor E on S such that $f^* D = nE$.

One issue: D may not pull back,
The next definition resolves this issue

<u>Def</u> A <u>generalized effective Cartier divisor</u> on a scheme X is a pair (L, s) where L is an invertible sheaf on X and $s \in L(X)$.

This corresponds with an actual divisor if and only if s is regular.

We can <u>always</u> pull these back.

Addition generalizes to
$$(L, s) \cdot (L', s') = (L \otimes L', s \otimes s')$$

Thus set of generalized Cartier divisors $/ \cong$ is a monoid

$\mathrm{Div}^+(X)$.

Consider the fibered category $\mathcal{D} \to Sch$ of pairs $(T, (L, s))$ where $(L, s) \in \mathrm{Div}^+(T)$. where the morphisms

$(T', (L', s')) \longrightarrow (T, (L, s))$

are pairs $(g, g^\flat)$

- $g : T' \longrightarrow T$
- $g^\flat : (L', s') \xrightarrow{\sim} (g^*L, g^*s)$

<u>Proposition</u> $\mathcal{D} \cong [\mathbb{A}^1/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts by multiplication

X scheme, $n \geq 1$ integer & $(L, s) \in \mathrm{Div}^+(X)$

Then define the root stack

$\mathfrak{X} = (X, \sqrt[n]{(L, s)})$ as the fiber product of

$$\begin{array}{ccc} & & [\mathbb{A}^1/\mathbb{G}_m] \\ & & \downarrow \\ X & \xrightarrow{(L,s)} & [\mathbb{A}^1/\mathbb{G}_m] \end{array}$$

By the proposition, a morphism $T \to \mathcal{X}$ from a scheme is a triple

$$(f: T \to X, \underset{\substack{\cap \\ \mathrm{Div}^+(T)}}{(M, t)}, \varphi)$$

$$\varphi: (M^{\otimes n}, t) \xrightarrow{\sim} (f^*L, f^*s)$$

unique if pullback is a divisor.

<u>Proposition</u>. $\mathcal{X}_n \to X$ is an isomorphism outside of the support of (L, s)

. $\mathcal{X}_n$ is DM if n is invertible on X.

Coarse moduli spaces

<u>Def</u> $\mathcal{X}/S$ algebraic stack.
A coarse moduli space for $\mathcal{X}$ is a morphism $\pi: \mathcal{X} \to X$ to an algebraic space over S such that

(i) π is initial for maps to algebraic spaces over S

$$\begin{tikzcd} \mathcal{X} \arrow[d] \arrow[dr] & \\ X \arrow[r, dashed, "\exists !"] & Y \end{tikzcd}$$ algebraic space

(ii) For an algebraically closed field k, the map

$|\mathcal{X}(k)| \longrightarrow X(k)$ is bijective

$\underbrace{|\mathcal{X}(k)|}_{\cong \text{ classes in } \mathcal{X}(k)}$

Theorem (Keel-Mori) $\mathcal{X}$ algebraic stack. Assume that the inertia $I \to \mathcal{X}$ is finite (such as when $\mathcal{X}$ is DM + separated), then $\mathcal{X}$ has a coarse moduli space. (Stacks tag 0DUT, many other sources as well)

Example. $\mathcal{M}_g$ has a coarse moduli space M_g.

- a root stack $\mathcal{X}_n$ has coarse moduli space X.

Remark. Root stacks are very important in the study of DM stacks. Less than a decade ago (~2015) Satriano and Geraschenko proved that over an algebraically closed field k, every tame DM stack (order of stabilisers coprime to char(k)) which is smooth and separated, is obtained by combining the root stack construction with the "canonical stack" (a "stacky resolution of singularities").