Session 4 Splitting of fibered categories / Groupoids

§1. Splitting of fibered categories:

Let $F \xrightarrow{\varphi} C$ be a fibered category.

- $\forall\, U \in C$, $F(U)$ a category with obj: $\xi \in F$ st $\phi(\xi) = U$

maps:
$$\begin{tikzcd} \xi \arrow[r] \arrow[d] & \delta \arrow[d] \\ U \arrow[r, "\mathrm{id}_U"'] & U \end{tikzcd}$$

- $\forall$
$$\begin{tikzcd} V \arrow[r] & U & \text{in } C \\ F(V) \ni \delta \arrow[u] \arrow[r] & \xi \in F(U) \arrow[u] \end{tikzcd}$$
st $\delta \to \xi$ is cartesian.

Want to think of F as a functor
$$\begin{aligned} C^{op} &\longrightarrow Cat \\ U &\longmapsto F(U) \end{aligned}$$

This is not a functor!

If $V \to U$ in C, do we get a map $F(U) \to F(V)$?

$$\begin{tikzcd} V \arrow[r, "f"] & U & \text{in } C \\ & \xi \in F(U) \arrow[u] \end{tikzcd}$$

Since $F \to C$ is a fibered category, $\exists$ $\underset{\in F(V)}{\delta} \to \xi$ cartesian lying above f

$$\begin{tikzcd} V \arrow[r, "f"] & U \\ \delta \arrow[u] \arrow[r] & \xi \arrow[u] \end{tikzcd}$$

$\rightsquigarrow$ Seeing F as a functor requires choosing Cartesian maps lying above maps in C.

Definition (Cleavage):

A cleavage for a fibered category is a choice of a Cartesian map for each map in C and each object in the target.

- Given a fibered category and a cleavage, a map $f: V \to U$ induces a functor:

$$f^* := F(f) : F(U) \to F(V)$$

$$\begin{tikzcd} \xi \arrow[r, mapsto] & V \arrow[r, "f"] & U \\ & f^*\xi \arrow[u] \arrow[r] & \xi \arrow[u] \end{tikzcd}$$

$$\delta \to \xi \longmapsto f^*\delta \to f^*\xi$$

- Is
$$\begin{aligned} C^{op} &\to Cat \\ U &\mapsto F(U) \\ f &\mapsto f^* \end{aligned}$$
a functor?

$$\begin{tikzcd} W \arrow[r, "g"] & V \arrow[r, "f"] & U \\ g^*f^*\xi \arrow[u] \arrow[r] & f^*\xi \arrow[u] \arrow[r] & \xi \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "f\circ g"] & U \\ (f\circ g)^*\xi \arrow[u] \arrow[r] & \xi \arrow[u] \end{tikzcd}$$

$\in F(W)$ $\qquad\qquad$ $\in F(W)$

$$g^*f^* \cong (f\circ g)^*$$

but $g^*f^* \neq (f\circ g)^*$ in general

Definition (Splitting Category)

Let $F \to C$ be a fibered category. A splitting of F is a subcategory $K \subset F$ st

1) every arrow in K is Cartesian.
2) $\forall$ $f: V \to U$ in C and $\forall$ $\xi \in F(U)$, $\exists!$ arrow $\delta \to \xi$ in K over f.

3) if $U \in C$ and $\xi \in F(U)$, then $\xi \xrightarrow{id_\xi} \xi$ in $F(U)$ is in K.

We write (F, K) a split.

Lemma

The category of split fibered categories that respects the splitting is equivalent to the category $\mathrm{Fun}(C^{op}, \mathrm{Cat})$.

Proof:

$(\Rightarrow)$
$$\begin{aligned} C^{op} &\longrightarrow \mathrm{Cat} \\ U &\longmapsto F(U) \\ f &\longmapsto f^* \end{aligned}$$
is a functor because:

$$\begin{tikzcd} U \arrow[r, "id_U"] & U \\ \xi \arrow[u] \arrow[r, "id_\xi"'] & \xi \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, "g"] & V \arrow[r, "f"] & U \\ (f\circ g)^*\xi \arrow[u] \arrow[rr] & & \xi \arrow[u] \end{tikzcd} \text{ in } C$$

$$(f \circ g)^* \xi = g^* f^* \xi$$

$(\Leftarrow)$ $F : C^{op} \longrightarrow \mathrm{Cat}$

Let F^{fib} be the fibered category

obj: (U, σ) with $U \in C$, $\sigma \in F(U)$
maps: $(V, \delta) \longrightarrow (U, \sigma)$ is a pair (g, α)

$$g : V \longrightarrow U \text{ in } C$$
$$\delta \xrightarrow{\alpha} F(g)(\sigma) \quad \text{in } F(V)$$

$$\begin{aligned} F^{fib} &\longrightarrow C \\ (U, \sigma) &\longmapsto U \end{aligned}$$

Let K be the subcategory of arrows of the form (g, id) with g an arrow in C.

(*)
$$\begin{tikzcd} & V \arrow[r, "f"] & U & \text{in } C \\ (V,\delta) \arrow[ur] \arrow[rr] & & (U,\sigma) \arrow[u] & \in F^{fib}(U) = K(U) \end{tikzcd}$$

$(V,\delta) \in F^{fib}(V)$

with $\delta \longrightarrow F(f)(\sigma)$

$\Rightarrow \delta = F(f)(\sigma) \xrightarrow{id} F(f)(\sigma)$ for a lift in K

$\Rightarrow (f, id)$ unique possible choice in K.

(*) K contains all identities:

$U \in C$ and $\sigma \in F(U)$

$(U,\sigma) \xrightarrow{(id,id)} (U,\sigma)$ is in K

$\sigma \longrightarrow F(id)(\sigma) = id(\sigma) = \sigma$ because F is a functor

(*) All arrows are Cartesian:

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[rr, bend left, "f"] & V \arrow[r, "g"] & U & \text{in } C \\ & (V,\delta) \arrow[u] \arrow[r, "{(g,id)}"] & (U,\sigma) \arrow[u] & \\ (W,\partial) \arrow[uu] \arrow[ur, dashed, "{(h,id)}"] \arrow[urr, "{(f,id)}"'] & & & \end{tikzcd}$$

$\delta = F(g)(\sigma)$

$\partial = F(f)(\sigma)$
$= F(g \circ h)(\sigma)$
$= F(h) \circ F(g)(\sigma)$ } F is a functor
$= F(h)(\delta)$

Examples:

1) $F : C^{op} \longrightarrow Set$

F^{fib} is a splitting fibered category.

2) G a group seen as a category with one object.

obj: $*_G$

maps: $*_G \xrightarrow[g \in G]{} *_G$ (elemts of G)

Let G and H be groups. A functor $f : G \to H$ is the same as gp homomorphism.

- $G \xrightarrow{f} H$ is fibered category iff f is surjective

$$\begin{tikzcd}
*G \arrow[r, "g \in G"] \arrow[d] & *G \arrow[d] & \text{in } G \arrow[d, "f"] \\
*H \arrow[r, "h \in H"'] & *H & \text{in } H
\end{tikzcd}$$

st $f(g) = h$

- All arrows are cartesian

$$\begin{tikzcd}
*G \arrow[r, dashed, "g_2 g_1^{-1}"'] \arrow[rr, bend left, "g_2 \in G"] & *G \arrow[r, "g_1 \in G"] \arrow[d] & *G \arrow[d] \\
& *H \arrow[r, "h_1 \in H"'] & *H \\
*H \arrow[uu, bend left] \arrow[ur, "h_3 = h_2 h_1^{-1} \in H"] \arrow[urr, bend right, "h_2 \in H"'] & &
\end{tikzcd}$$

- What is a splitting here ?

Assume $f : G \to H$ surjective so that $G \to H$ is a fibered category.

a splitting = a section $H \xrightarrow{s} G$ which is gp homoph[i]sm.

To get a non-example, consider any group extension G of H by K that doesn't split.

Theorem :

Let $F \to C$ be a fibered category.

There exists a split fibered category $(\hat{F}, K)$ st $\hat{F}$ is equivalent to F.

Proof :

Let $\qquad H : C^{op} \to Cat$

$$U \mapsto \mathrm{HOM}_C(C/_U, F)$$

$H^{fib} = \hat{F}$

Let $\hat{F}$ be the split fibered category associated to the functor H.

$\tilde{F}$: obj : (U, σ) with $U \in C$ and $\sigma \in H(U) = \mathrm{HOM}(C/_U, F)$

maps : $(V, \delta) \xrightarrow[(g,\alpha)]{} (U, \sigma)$ $\qquad g : V \to U$ in C

α base preserving nat. transf

Show that $\tilde{F}$ and F are equivalent.

$$\tilde{F} \xrightarrow{e} F$$

$$(U, \sigma) \mapsto \sigma(id_U)$$

$$(g, \alpha) \mapsto \sigma(g) \circ \alpha(id_V)$$

Check that :

- it is a morphism of fibered categories
- e restricts to the evaluation map on the fibers

$\Rightarrow$ 2-Yoneda lemma, e is an equivalence on each fiber $\Rightarrow$ e an equivalence

∎

Remark :

$$\mathrm{Fun}(C^{op}, \mathrm{Cat}) \hookrightarrow \underset{\text{over } C}{\text{Category of fibered categories}}$$

is faithful + essentially surjective but not full.

§ 2. Groupoids / Categories fibered in groupoids :

Definition (Groupoid)

A groupoid is a category where all morphisms are isomorphisms.

Examples :

1) Any set seen as a category.

2) G a group seen as a category with one object.

Definition (Category fibered in groupoids)

- A category fibered in groupoids over a category C is a fibered category $F \to C$ st $\forall U \in C$, $F(U)$ is a groupoid.

Example :

$G \xrightarrow{f} H$ surjective gp homomorphism.

it is a category fibered in groupoids.

Proposition

let F and F' be fibered categories in groupoids over C.

Then, $\mathrm{HOM}_C(F, F')$ is a groupoid.

Proof:

Let $f, g : F \to F'$ morphisms of fibered categories and $\xi : f \to g$ a morphism. Is it an isomorphism?

For $u \in F_1$ $\xi_u : f(u) \to g(u)$ an arrow in F'

Let $X \in C$ the image of u via $p_F : F \to C$

ξ_u base-pres.

$$\begin{tikzcd} f(u) \arrow[r, "\xi_u"] \arrow[d] & g(u) \arrow[d] & \text{in } F' \\ X \arrow[r, "id_X"'] & X & \text{in } C \end{tikzcd}$$

hence ξ_u is an arrow in $F'(X)$ which is a groupoid. ∎

Definition (fiber product of groupoids)

$$\begin{tikzcd} & \mathcal{G}_1 \arrow[d, "f"] \\ \mathcal{G}_2 \arrow[r, "g"'] & \mathcal{G} \end{tikzcd}$$

f, g functors

a diagram of groupoids. We define a groupoid $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$ as follows:

obj: triples (x, y, σ) where $x \in \mathcal{G}_1$, $y \in \mathcal{G}_2$ and σ is a morphism $f(x) \to g(x)$ in $\mathcal{G}$.

maps: $(x', y', \sigma') \to (x, y, \sigma)$ is a pair of isomorphisms $x' \xrightarrow{a} x$ $y' \xrightarrow{b} y$ st

$$\begin{tikzcd} f(x') \arrow[r] \arrow[d, "f(a)"'] \arrow[dr, phantom, "\circlearrowright"] & g(x') \arrow[d, "f(b)"] \\ f(x) \arrow[r] & g(x) \end{tikzcd}$$

commutes.

check it is a groupoid.

- There are functors $p_j : \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 \to \mathcal{G}_j$ and a natural isomorphism of functors $\Sigma : f \circ p_1 \to g \circ p_2$

$$\begin{tikzcd}
\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{G}_2 \arrow[d, "f"] \\
\mathcal{G}_1 \arrow[r, "g"'] & \mathcal{G}
\end{tikzcd}$$

The category $\mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2$, together with the functors p_1 and p_2 and Σ have the following universal property:

If $\mathcal{H}$ is another groupoid and

$$\begin{tikzcd}
\mathcal{H} \arrow[dr, dashed, "h"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2 \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathcal{G}_2 \arrow[d, "f"] \\
& \mathcal{G}_1 \arrow[r, "g"'] & \mathcal{G}
\end{tikzcd}$$

and $\gamma : f \circ \beta \to g \circ \alpha$ an isomorphism of functors

Then, there exists $(h : \mathcal{H} \to \mathcal{G}_1 \times_{\mathcal{G}} \mathcal{G}_2, \, \partial_1, \partial_2)$

where h is a functor

$\left.\begin{array}{l} \partial_1 : \alpha \to p_1 \circ h \\ \partial_2 : \beta \to p_2 \circ h \end{array}\right\}$ Iso of functors

and

$$\begin{tikzcd}
f \circ \alpha \arrow[r, "f(\partial_1)"] \arrow[d, "\gamma"'] & f \circ p_1 \circ h \arrow[d, "\Sigma \circ h"] \\
g \circ \beta \arrow[r] & g \circ p_2 \circ h
\end{tikzcd}$$

The data $(h, \lambda_1, \lambda_2)$ is unique up to a unique isomorphism.

- Let C be a category and

$$\begin{tikzcd} & F_1 \arrow[d, "c"] \\ F_2 \arrow[r, "d"'] & F_3 \end{tikzcd}$$

a diagram of fibered categories in groupoids over C.

- Consider G a category fibered in groupoids over C, with

$$\left.\begin{array}{l} \alpha : G \to F_1 \\ \beta : G \to F_2 \end{array}\right\} \text{ morphisms of f.c}$$

$$c \circ \alpha \xrightarrow{\gamma} d \circ \beta \quad \text{an isomorphism of fibered cat } G \to F_3$$

Giving the data (α, β, γ) is equivalent to giving an object of

$$HOM_C(G, F_1) \times_{HOM_C(G, F)} HOM_C(G, F_2)$$

Proposition :

$:=$ fiber product category

There exists a collection of data $(G, \alpha, \beta, \gamma)$ as above st for every H category fibered in groupoids over C,

$$HOM_C(H, G) \longrightarrow HOM(H, F_1) \times_{HOM(H, F_3)} HOM(H, F_2)$$

$$h \longmapsto (\alpha \circ h, \beta \circ h, \gamma \circ h)$$

is an isomorphism

(IV)

Let G be a groupoid. We can describe it as follows:

- G_0 set of object
- for each objects x, y, a set $G(x,y)$ of morphisms from x to y.
- for every object x, a designated element id_x in $G(x,x)$.
- for each triple objects x, y and z, a functions

comp: $G(y,z) \times G(x,y) \to G(x,z)$
$(g, f) \mapsto g \circ f$

- for each object x, y, an inverse function

$$G(x,y) \to G(y,x) : \underset{x \to y}{f} \mapsto \underset{y \to x}{f^{-1}}$$

All satisfying, for any $f: x \to y$, $g: y \to z$, $h: z \to w$:

$$f\, id_x = f \quad \text{and} \quad id_y\, f = f$$

$$(hg) f = h (gf)$$

$$f f^{-1} = id_y \quad \text{and} \quad f^{-1} f = id_x$$

If $f \in G(x,y)$, x is called source of f and y target of f.
We write $x = s(f)$
$y = t(f)$

Let C be a category with finite fiber products.

A groupoid object in C consists of:

- A pair (U, R) of objects
  - obj: U, arrows: R
- Five morphisms:

$$s, t : R \to U \ , \quad e : U \to R \ , \quad i : R \to R$$

source, target, unit, inverse

$$m : R \times_{s,U,t} R \to R :$$

multipl.

satisfying the following:

1)

$$\begin{tikzcd} U \arrow[r, "e"] \arrow[d, "\varepsilon"'] \arrow[dr, "id_U"] & R \arrow[d, "t"] \\ R \arrow[r, "s"'] & U \end{tikzcd} \qquad \begin{tikzcd} R \arrow[d, "s"'] \arrow[dr, phantom, "\circlearrowright"] & R \times_{s,U,t} R \arrow[l, "p_1"'] \arrow[r, "p_2"] \arrow[d, "m"] \arrow[dr, phantom, "\circlearrowright"] & R \arrow[d, "t"] \\ U & R \arrow[l, "s"] \arrow[r, "t"'] & U \end{tikzcd}$$

(Unit)

$$\begin{tikzcd} U \times_{id,U,t} R \arrow[d, "e \times id"'] \arrow[r, equal] & R \arrow[d, equal] \arrow[r, equal] & R \times_{s,U,id} U \arrow[d, "id \times e"] \\ R \times_{s,U,t} R \arrow[r, "m"'] & R & R \times_{s,U,t} R \arrow[l, "m"] \end{tikzcd}$$

(inverse)

$$\begin{tikzcd} R \arrow[r, "i \times id"] \arrow[d, "s"'] \arrow[dr, phantom, "\circlearrowright"] & R \times_{s,U,t} R \arrow[d, "m"] \arrow[dr, phantom, "\circlearrowright"] & R \arrow[l, "id \times i"'] \arrow[d, "t"] \\ U \arrow[r, "e"'] & R & U \arrow[l, "e"] \end{tikzcd}$$

(associativity)

$$\begin{tikzcd} R \underset{s,U,t}{\times} R \underset{s,U,t}{\times} R \arrow[r, "m\times id"] \arrow[d, "id\times m"'] \arrow[dr, phantom, "\circlearrowright"] & R \underset{s,U,t}{\times} R \arrow[d, "m"] \\ R \underset{s,U,t}{\times} R \arrow[r, "m"'] & R \end{tikzcd}$$

Examples:

1) A groupoid object in the category of sets (Set) is a groupoid in the usual sense.

U = the set of all objects in Set

R = the set of all arrows in Set

$s(a \to b) = a \qquad t(a \to b) = b$

$e(a) = id_a \qquad i(\underset{\in R}{f : a \to b}) = \underset{\in R}{f^{-1} : b \to a}$

$m(f, g) = g \circ f$.

3) Let S be a scheme and G/S a group scheme acting on an S-scheme X. Define the action groupoid associated to this G-action as:

(i) $U = X$, $R = X \times_S G$

(ii) $s : X \times_S G \to X$ is the first projection.

$t : X \times_S G \to X$ is the action map.

(iii) $e : X \to X \times_S G$ is induced by $S \to G$ section.

(iv) $i : X \times_S G \to X \times_S G$, $(x, g) \mapsto (g.x, g^{-1})$

(v) $$X \times_S G \underset{X}{\times} X \times_S G \xrightarrow{m} X \times_S G$$

$$\wr\!\wr$$

$$X \times_S G \times_S G$$

is induced by $G \times_S G \to G$ the law.

Then we get a groupoid object in (Sch/S) usually denoted $\{X/G\}$.

<u>Discussion</u>: difference between a groupoid object in C and a group object in C.