# Stacks



. Recall on descents:

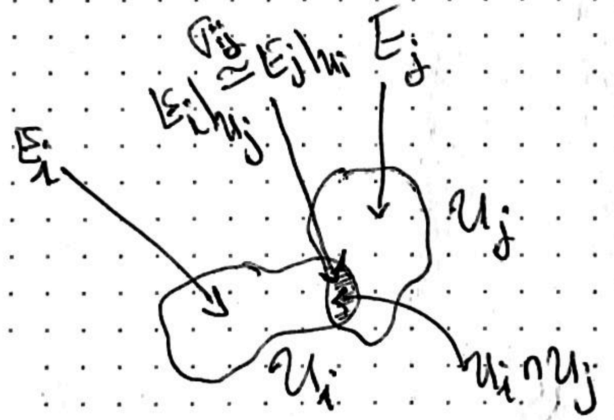
The idea: let $X$ be a Topological space.

$Op(X)$ = the category of open sub sets

we have the following equivalence of categories:

$$\left\{ \begin{array}{c} E \\ \downarrow \\ X \end{array} \begin{array}{l} \text{Vector bundles} \\ \text{on } X \text{ of finite} \\ \text{rank} \end{array} \right\} \xrightarrow[\text{restriction maps}]{} \left\{ \begin{array}{c} E_i \\ \downarrow \\ U_i \end{array} \begin{array}{l} U_i \underset{open}{\subseteq} X \\ E_i \text{ of rank } n \text{ v.b.} \\ \& \; E_i|_{U_j} \underset{\varphi_{ij}}{\simeq} E_j|_{U_i} \\ \varphi_{ij} \circ \varphi_{jk} = \varphi_{ik} \end{array} \right\}$$

for any covering: $X = \bigcup_{i \in I} U_i$




we can see

$$\begin{tikzcd} & E_i \arrow[d] \\ U_i \cap U_j = U_i \times_X U_j \arrow[r, "p_{r_2}"] \arrow[d, "p_{r_1}"'] & U_i \arrow[d, hook, "i"] \\ E_j \arrow[r] & U_j \arrow[r, hook] & X \end{tikzcd}$$

$p_{r_2}^*(E_i) \xrightarrow[\sim]{\varphi_{ij}} p_{r_1}^*(E_j)$

over $U_j \cap U_i$

All what we need to such gluing data is ~~a covering~~ over any category $\mathcal{C}$ is a covering data given by a grothendieck Top over $\mathcal{C}$.

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Let $\mathcal{C}$ be a site, with finite ~~fiber~~ product

$p : F \longrightarrow \mathcal{C}$ be a fiber category

( F muste be seen as objects we want to study

more presisly for $\underline{X \in \mathcal{C}}$

$F(X)$ muste be seen an objects over $X$ )

exp: $\mathcal{C} = Sch_{zariski}$ with Zariski Top )

$X \in \mathcal{C}$ a scheme

$F(X) =$ the category of vector bundles on $X$

so $F = \{$ the Category of vector bundles over schemes $\}$

**Def.** A covering $\{X_i \to X\}$ is of effective descent of F

if: the natural functor:

$$\varepsilon : F(X) \longrightarrow F(\{X_i \to X\})$$

$$E \longmapsto (\{E_i, \sigma_{ij}\}) \quad \sigma_{ij} \text{ is } \underline{\text{cano}}$$

is a equivalence of cat.

**Def.** ~~Let object~~ $E = (\{E_i, \sigma_{ij}\}) \in F(\{X_i \to X\})$ an object

$\{\sigma_{ij}\}$ is effective if $E$ is in the essential image of $\varepsilon$



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Very important lemma:

**Lemma:** let $\mathcal{C}$ a site with coproduct, suppose that product and coproduct commutes.

Assume that for any objects $\{X_i\}_{i\in I}$ in $C$

the Natural functor: $F(\coprod_i X_i) \longrightarrow \prod_i F(X_i)$

is an equivalence of category.

Set $Q = \coprod_{i\in I} X_i$, then

$$\left\{\begin{array}{c} F(Y) \longrightarrow F(\{X_i \to Y\}) \\ \text{is an equivalence} \\ \text{of cat} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} F(Y) \longrightarrow F(Q \to Y) \\ \text{is an equivalence} \\ \text{of category} \end{array}\right\}$$

$\lrcorner$

# II) Examples of descents.

## Exp 1) Descent of closed sub schemes

let $Y$ be a scheme, $f: X \to Y$ an fppf morphisme.

the Natural diagrame

$$\begin{tikzcd} X\times_Y X \arrow[r, "pr_2"] \arrow[d, "pr_1"'] \arrow[dr, "g"] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \qquad g := f\circ pr_1 = f\circ pr_2$$

let $Z \hookrightarrow X$ be a closed sub scheme

$pr_i^*(Z) := Z \times_{(pr_i, X)} (X\times_Y X)$ the pullbacks.

**Proposition:**

$$\left\{\begin{array}{c} W \hookrightarrow Y \\ \text{closed} \end{array}\right\} \longrightarrow \left\{\begin{array}{c} Z \hookrightarrow X \\ \text{closed} \end{array} \middle/ pr_1^* Z = pr_2^* Z\right\}$$

$$W \longmapsto f^{-1}(W)$$

is an equivalence of cat.

Proof, we have the following correspondance

$$\left\{\begin{matrix} Z \subset X \\ \text{closed sub-schemes}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix} I \subset \mathcal{O}_X \\ \text{quasicoherent sheaves} \\ \text{of ideals}\end{matrix}\right\}$$

$$Z \longmapsto \ker(\mathcal{O}_X \xrightarrow{\text{rest}} \mathcal{O}_Z)$$

So, it suffices to show that the pullback map

$$\left\{\begin{matrix} \text{quasi-coherent} \\ \text{sheaves of ideals} \\ I \subset \mathcal{O}_Y\end{matrix}\right\} \longrightarrow \left\{\begin{matrix} \text{quasi-coherent sheaves} \\ \text{of ideals } I \subset \mathcal{O}_X \text{ s.t} \\ pr_1^* I = pr_2^* I\end{matrix}\right\}$$

$$pr_i : X \times_Y X \rightrightarrows X \xrightarrow{f} Y$$

is a bijection:

lemma (Theorem 4.3.12) : fppf-covering of schemes are effectif descent morphisms for Quasi Coherent Sheaves

Exp 2) Descent for open immbeddings.

denote $\mathcal{O}p = \left\{\begin{matrix} \text{obj } (X, U) \ / X \in Sch \ , U \underset{\text{open}}{\subseteq} X \\ Hom_{\mathcal{O}p}((X,U),(Y,V)) = \{ f \in Hom_{Sch}(X,Y) \text{ s.t } f^{-1}(V) \supset U \}\end{matrix}\right.$

let $p : \mathcal{O}p \longrightarrow Sch$ makes $\mathcal{O}p$ a fibred cate over Sch.
$(X, U) \longmapsto X$

**Proposition:** Any fppf covering $f: S' \to S$ is an effective descent morphism for $\mathcal{P}$.

**Proof:**

$$\begin{tikzcd} X = S' \times_S S' \arrow[r, "pr_1"] \arrow[d, "pr_2"'] & S' \arrow[d, "f"] \\ S' \arrow[r, "f"'] & S \end{tikzcd}$$

we have to show that for $U' \underset{open}{\subseteq} S'$ s.t

$$pr_1^{-1}(U') = pr_2^{-1}(U') \quad \text{in } X.$$

$$\underset{?}{\Longrightarrow} \quad U' = f^{-1}(U) \quad \text{for } U \underset{open}{\subseteq} S$$

- If $U$ exists is unique as fppf maps are surj.
- Proposition (1.1.5) fppf map are open
  by this proposition set $U := f(U') \underset{open}{\subseteq} S$ : (no descent conditions)
- Now have $f^{-1}(U) \supset U'$.

let $x \in f^{-1}(U) \setminus U'$, then there exists a point $y \in U'$ such that $x, y$ have the same image in $U$

So $(x, y) \in X = S' \times_S S'$ which is in $pr_1^{-1}(U')$
but not in $pr_2^{-1}(U')$

Contradiction, Hence $f^{-1}(U) = U'$. $\square$

$\boxed{\text{Exp 3}}$ Descent for Affine morphismes.

Set Aff the category of affine morphismes of schemes

$$\mathrm{Aff} = \{ f : X \to Y \mid f \text{ affine} \}$$

$$\Big\Vert$$

$$\left[ f^{-1}\left(\begin{matrix}\text{Affine}\\ \text{open}\\ \text{in } Y\end{matrix}\right) \text{ is } \begin{matrix}\text{affine}\\ \text{open in } X\end{matrix} \right]$$

$$\mathrm{Hom}_{\mathrm{Aff}}\left( \begin{matrix} X \\ f\downarrow \\ Y \end{matrix} , \begin{matrix} X' \\ f'\downarrow \\ Y' \end{matrix} \right) = \left\{ \begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "\beta"'] & Y' \end{tikzcd} , \text{ commute} \right\}$$

let $p : \mathrm{Aff} \longrightarrow \mathrm{Sch}$, makes Affine in a fibred category
$$(X \xrightarrow{f} Y) \longmapsto Y$$

Proposition: For any $s : S' \longrightarrow S$ an fppf covering
$s$ is an effective descent map for Aff.

Proof: Strategy rephrase this as a statement about quasi-coherent sheaves of algebras.

Let Alg = Category of pairs $(Y, \mathcal{A}) \,/\, Y \in \mathrm{Sch}$, $\mathcal{A}$ quasi-coh. $\mathcal{O}_Y$-Alg on $Y$

$$\mathrm{Hom}_{\mathrm{Alg}}\big((Y', \mathcal{A}'), (Y, \mathcal{A})\big) = \left\{ (h, \varepsilon) \,\middle/\, \begin{matrix} h \in \mathrm{Hom}_{\mathrm{Sch}}(Y', Y), \\ \varepsilon : h^*(\mathcal{A}) \longrightarrow \mathcal{A}' \\ \text{morphisme of quasi-coherent } \mathcal{O}_Y\text{-Alg} \end{matrix} \right\}$$

we have $q : \mathrm{Alg} \longrightarrow \mathrm{Sch}$, makes Alg into a fibred category over Sch.
$$(Y, \mathcal{A}) \longmapsto Y$$

~~By~~ Natural functor

$$\mathrm{Alg} \longrightarrow \mathrm{Aff}$$

$$(Y, \mathcal{A}) \longmapsto \left(\mathrm{Spec}_Y(\mathcal{A}) \longrightarrow Y\right)$$

quasi-coherent $\mathcal{O}_Y$-Alg.

(by construction affine map)

**Fact:** this is an equivalence of fibred categories over Sch.

~~by the previous~~ Therefore we must show the descent for fppf on Alg.

We recall that fppf-coverings are effective descents for quasi-coherent sheaves over Sch, In particular for alg-objects $\square$

$\boxed{\text{Exp } 4)}$ Descent for polarized schemes:

$$\text{let } \mathrm{Pol} = \left\{ (f: X \to Y, L) \;\middle/\; \begin{array}{l} X, Y \in \mathrm{Sch} \\ f \in \mathrm{Hom}_{\mathrm{Sch}}(X, Y) \\ L \in \mathrm{Pic}(X/Y) \end{array} \right\}$$

(proper flat)

(relatively ample)

morphism $(f': X' \to Y', L') \longrightarrow (f: X \to Y, L)$ is a triple

$$\begin{tikzcd} X' \arrow[r, "b"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ Y' \arrow[r, "a"] & Y \end{tikzcd}$$

and $\varepsilon : b^*(L) \longrightarrow L'$ an isom

We have a functor

$$p: \mathrm{Pol} \longrightarrow \mathrm{sch}, \quad \text{is a fibred cat}$$
$$(f: X \to Y, L) \longmapsto Y.$$

<u>Proposition</u>: fppf $f: S' \to S$ morphism is an effective descent morphism for. Pol.

<u>Proof</u>: (<u>exercise</u>)

exp. <u>Descent for $\mathcal{M}_g$, $g \geq 2$</u>:

<u>Recall</u>. $\mathcal{M}_g(S)$ = category of $\begin{array}{c} \mathcal{C} \\ \pi \downarrow \\ S \end{array}$ smooth proper maps

geometric fibers are connected genus $g$ curves
morphisms are isomorphic curves over. $S$

$$p: \mathcal{M}_g \longrightarrow \mathrm{sch} \quad \text{is a fibred category in groupoids}$$
$$(\mathcal{C} \xrightarrow{\pi} S) \longmapsto S$$

Let $X \xrightarrow{f} Y$ be a morphism, so $\mathcal{M}_g(X \xrightarrow{f} Y)$ is the category of pairs $(C_X, \sigma)$ / $C_X \in \mathcal{M}_g(X)$ and

$\sigma$ is an isomorphism. $\sigma: p_2^* C_X \xrightarrow{\sim} p_1^* C_X$.

with the natural compatibility conditions

Morphisms

$$(C'_X, \sigma') \xrightarrow{\varepsilon} (C_X, \sigma)$$

is a morphism: $a: C'_X \longrightarrow C_X$ such that the following diagram commute

$$\begin{tikzcd} p_2^* C'_X \arrow[r, "p_2^*\varepsilon"] \arrow[d, "\sigma'"'] & p_2^* C_X \arrow[d] \\ p_1^* C'_X \arrow[r, "p_1^*\varepsilon"'] & p_1^* C_X \end{tikzcd}$$

Proposition: If $g \geq 2$, $f: X \to Y$ quasi-compact fppf map

then the pullback

$$f^*: \mathcal{M}_g(Y) \longrightarrow \mathcal{M}_g(X \xrightarrow{f} Y)$$

$$C \longmapsto (f^*C, \mathrm{can})$$

is an equivalence of categories.

Proof: Recall the following.

lemma: for $g \geq 2$, for any $(\pi : C \to S) \in \mathcal{M}_g(S)$ the sheaf $\Omega^{\otimes 3}_{C/S}$ is relatively ample

We have a functor for $g \geq 2$

$$\varphi : \mathcal{M}_g \longrightarrow \mathrm{Pol}$$
$$(\pi : C \to S) \longmapsto \left(C \xrightarrow[\pi]{} S,\ \Omega^1_{C/S}\right)$$

We have the following commutative diagram for $S' \to S$ fppf

$$\begin{tikzcd} \mathcal{M}_g(S) \arrow[r, "\text{essential surjective}"] \arrow[d] & \mathrm{Pol}(S) \arrow[d, "\simeq\ \text{eq.}"] \\ \mathcal{M}_g(S' \to S) \arrow[r, "\text{essential surjective}"'] & \mathrm{Pol}(S' \to S) \end{tikzcd}$$

$\Rightarrow \mathcal{M}_g(S) \longrightarrow \mathcal{M}_g(S' \to S)$ ~~is essentially surjective~~

by glueing morphisms under fppf-map the functor above is fully-faithfull.

Essential surjectivity need work page 34

III) Stacks : Let $C$ be a site.

Def: A fibred category in groupoids $\boxed{p : F \longrightarrow C}$ is a stack if for every $X \in C$ and a covering $\{X_i \to X\}_I$ the functor

$$F(X) \longrightarrow F(\{X_i \to X\})$$

is an equivalence of catories.

Recall that if $C$ has finite coproducts and that they commute with products such that $F(\coprod_i X_i) \longrightarrow \prod_i F(X_i)$ is an equivalence of cate

we have: $F(X) \to F(\{X_i \to X\})$ equivalence of cat

$\Leftrightarrow F(X) \to F(Q \to X)$ e //

where $Q = \coprod_i X_i$

Some properties.

Def: Let $F$ be a fibred category over $C$ with some choice of a cleavage, let $x, y \in F(U)$

set the presheaf: $\underset{F(U)}{Hom}(x,y) : (C/U)^{op} \longrightarrow$ Groupoids

$$(f : V \to U) \longmapsto \underset{F(V)}{Hom}(f^*(x), f^*(y))$$

if $(g : W \to V) \mapsto \underset{F(V)}{Hom}(f^*x, f^*y) \xrightarrow{g^*} \underset{F(W)}{Hom}(g^*f^*x, g^*f^*y)$
a morphisme over $U$

$$\| can$$

$$\underset{F(W)}{Hom}((fg)^*x, (fg)^*y)$$

NB: This definition forces $\underline{Hom}(x,y)$ to respect composition even if the cleavage you choose is not a splitting. Moreover different cleavage give canonically isomorphic presheaves.

Lemma: The following are equivalent.

(A) $\forall U \in C, \forall x,y \in F(U)$, $\underline{Hom}(x,y)$ is a sheaf.

(B) $\forall$ covering $V \to U$, $F(U) \to F(V \to U)$ is fully faithful.

Proof: Observation: Let $f: V \to U$ be a covering, and let $g = f \circ p_1 = f \circ p_2 : V \times_U V \to U$.
Then consider the following sequence.

$$\begin{array}{ccccc} \underline{Hom}(x,y)(U) & \to & \underline{Hom}(x,y)(V) & \rightrightarrows & \underline{Hom}(x,y)(V\times_U V) \\ \| & & \| & & \| \\ Hom_{F(U)}(x,y) & \to & Hom_{F(V)}(f^*(x), f^*(y)) & \rightrightarrows & Hom_{F(V\times_U V)}(g^* x, g^* y) \end{array}$$

An element $\phi \in \underline{Hom}(x,y)(V)$ whose restriction along the two projections are equal is exactly a morphisme
$$(f^* x, can) \to (f^* y, can) \text{ in } F(V \to U)$$

Fact: The exactness of the above sequence $\Leftrightarrow$ To full faithfulness of the funct $F(U) \to F(V \to U)$