Session 7 (Algebraic spaces)

I) Properties of sheaves

The goal of this section is to extend properties of objects (and morphisms) to sheaves on the corresponding site (and morphisms of sheaves).

Fix a site C.

Definition: (Stable property of objects)

- A class of objects $S \subseteq C$ is stable if for every covering $\{U_i \to U\}$, $U \in S$ iff $U_i \in S$.

- We call a property P of objects stable if the class of objects satisfying P is stable.

Eg: X a scheme. $C = (Sch/X)_{zar}$

$P =$ locally of finite type

locally of finite presentation

$S =$ "class of schemes over X locally of finite type"

Take $Y \to X$ in C, let $Y = \bigcup Y_i$ a Zariski cover

$\underbrace{Y \text{ is locally of ft}}_{Y \in S}$ iff $\underbrace{Y_i \text{ is locally of ft}}_{Y_i \in S \ \forall i}$.

Definitions:

(i) A closed subcategory of C is a subcategory $D \subseteq C$ st:

1) D contains all isomorphisms.
2) For all Cartesian diagram in C,

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

for which $f \in D$, $f' \in D$.

(ii) A closed subcategory $D \subseteq C$ is stable if $\forall f: X \to Y$ in C and every covering $\{Y_i \to Y\}$, the morphism f is in D iff the maps $f_i: X \times_Y Y_i \to Y_i$ are in D.

($\Rightarrow$ is contained in "closed subcat")

(iii) A stable closed category $D \subseteq C$ is local in domain if for all $f: X \to Y$ in C and all $\{X_i \xrightarrow{x_i} X\}$, $f \in D$ iff $f \circ x_i \in D$.

(iv) Let P be a property of morphisms in C satisfied by isomorphisms and closed under composition. Let D_P be the subcategory of C with same objects and whose morphisms satisfy P. We say that P

is stable (resp. local domain) property of morphisms if $D_P \subseteq C$ is stable (resp. local domain).

Eg: $C = (Sch/S)_{ét}$

(i) Stable propties in C: proper, separated, quasi-compact, surjective...

(ii) stable + local domain: locally of f.t / f.p, flat, étale, smooth...

in particular, to check that a morphism $X \to Y$ is "étale" for example, it is enough to check it locally on X or Y.

Definition:

S a scheme. Let $f: F \to G$ be a morphism of sheaves on $(Sch/S)_{ét}$.

(i) f is representable by schemes if for all S-scheme T and $\underset{h_T''}{T} \to G$, $F \times_G T$ is a scheme.

(ii) Let P be a stable property of morphisms of schemes. If f is representable by schemes, we say that f has property P if for every S-scheme T, the morphism of schemes $F \times_G T \xrightarrow{pr_2} T$ has P.

Example:

Let F and G be representable sheaves. Then any morphism $f : F \to G$ is representable by schemes.

Proof:

Say X represents F and Y represent G.

$T \to S$ a scheme and $T \to G$.

Then $T \times_G F = h_T \times_{h_Y} h_X \simeq h_{T \times_Y X}$ is represented by $T \times_Y X$.

$\blacksquare$

Lemma:

S a scheme.

$f : X \to Y$ a morphism of S-schemes.

If P is stable property of morphisms of schemes, then f has $P \Rightarrow h_X \to h_Y$ has P in the sense of the previous definition (ii).

Proof

$T \to S$ a scheme and $T \to Y$.

$h_X \times_{h_Y} T \longrightarrow T$ has P because f has P

$\| \quad$ represented by $\qquad$ and P is stable.

$X \times_Y T$

$\blacksquare$

Lemma

S a scheme.

F a sheaf on $(Sch/S)_{ét}$

Suppose that the diagonal $\Delta : F \to F \times_S F$ is representable by schemes. Then if T is an S-scheme, any morphism $f : T \to F$ is representable by schemes.

Proof:

Let $T' \xrightarrow{g} S$ be a scheme and $g : T' \to F$ a morphism.

We want to show that $T \times_F T'$ is representable by a scheme.

$$\begin{tikzcd} & T' \arrow[d, "g"] \\ T \arrow[r, "f"] & F \end{tikzcd} \quad \text{is isomorphic to} \quad \begin{tikzcd} & T \times_S T' \arrow[d, "f \times g"] \\ F \arrow[r, "\Delta"] & F \times_S F \end{tikzcd}$$

$f(t) = g(t)$ in F

and $F \times_{\Delta, F \times_S F} \underbrace{T \times_S T'}_{\text{scheme}}$ is representable by a scheme as Δ is representable by schemes.

$\blacksquare$

Definition (algebraic space)

Let S be a scheme.

An algebraic space over S is a functor $X : (Sch/S)_{ét}^{op} \to Set$ st:

(i) X is a sheaf for the étale topology.

(ii) $\Delta : X \to X \times_S X$ is representable by schemes.

(iii) there exists an S-scheme U and a surjective étale morphism $U \to X$.

A morphism of algebraic spaces is a morphism of functors.

Example:

A scheme is an algebraic space.

Rmk:

(i) if you replace étale topology by Zariski topology, you get a scheme (X is represented by U).

(ii) if you replace étale topology by fppf topology, you get an equivalent category of algebraic spaces.

(iii) Heuristicly, an algebraic space is gluing of affine schemes over the étale topology.

- Let $g : S' \to S$ be a morphism of schemes and let $(AS/_S)$ (resp. $(AS/_{S'})$) be the category of algebraic spaces over S (resp. S').

Define the category $\mathcal{C}$:

obj : pairs (X, f)

X : alg. space over S

$f : X \to S'$ morphism of S-alg. spaces.

arrows : $(X', f') \longrightarrow (X, f)$

is

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr, "f'"'] & & X \arrow[dl, "f"] \\ & S' & \end{tikzcd}$$

- Let $Y \in (AS/S')$.
- Define $Y_S : (Sch/S)^{op}_{ét} \longrightarrow Set$

$$T \mapsto (\varepsilon, y)$$

$\varepsilon : T \to S'$ an S-morphism
$y \in Y(\varepsilon)$

There is a natural morphism of functors :

$$f_Y : Y_S \longrightarrow S' (= h_{S'})$$

$$(\varepsilon, y) \mapsto \varepsilon$$

Proposition:

(1) Y_S is an algebraic space over S.

(2) The induced functor:

$$(AS/S') \longrightarrow \mathcal{C}$$
$$Y \longmapsto (Y_S, f_Y)$$

Proof:

(1) (i) Y_S is an étale sheaf over S:

Let $T \to S$ a scheme and $\{T_i \to T\}$ an étale cover. Let $\{(\varepsilon_i, y_i)\} \in Y_S(T_i)$ an element of the equalizer:

$$\prod_i Y_S(T_i) \rightrightarrows \prod_{ij} Y_S(T_i \times T_j)$$

we need to show that this collection is induced by a unique element $(\varepsilon, y) \in Y_S(T)$.

- $\varepsilon_i : T_i \to S'$ i.e $\varepsilon_i \in S'(T_i)$

then $\{\varepsilon_i\}_i$ defines an element of the equalizer

$$\prod_i S'(T_i) \rightrightarrows \prod_{i,j} S'(T_i \times T_j)$$

hence is induced by a unique element $\varepsilon \in S'(T)$ since S' is a sheaf.

View T and $\{T_i\}$ as S'-schemes via ε and $\{\varepsilon_i\}_i$.

$\{y_i\}_i$ is given by the equalizer

$$\prod_i Y(T_i) \rightrightarrows \prod_{i,j} Y(T_i \times T_j)$$

hence is induced by a unique $y \in Y(\varepsilon)$

$\leadsto (\varepsilon, y) \in Y_S(T)$ is the unique element induced by the collection $\{(\varepsilon_i, y_i)\}$.

- $Y_S \xrightarrow{\Delta} Y_S \times Y_S$ is representable

First note that the fiber product

$$\begin{tikzcd} & Y_S \times_S Y_S \arrow[d, "f_Y \times f_Y"] \\ S' \arrow[r, "\Delta_{S'/S}"'] & S' \times_S S' \end{tikzcd}$$

is isomorphic to $Y \times_{S'} Y$ ⊛ in (Sch / S')

Let $T \to S$ a scheme and $g : T \longrightarrow Y_S \times_S Y_S$

Then the fiber product $L := T \times_{g, Y_S \times_S Y_S, \Delta} Y_S$ is given by

taking first the fiber product T':

$$\begin{tikzcd} & Y_S \times Y_S \arrow[d, "f_Y \times f_Y"] \\ T \arrow[ur, "g"] \arrow[r] & S' \times S' \\ T' \arrow[u] \arrow[r] & S' \arrow[u, "\Delta_{S'/S}"'] \end{tikzcd}$$

and then
$$\begin{tikzcd} T' \arrow[r] & Y \times_{S'} Y \\ & Y \arrow[u, "\Delta_Y"'] \end{tikzcd}$$

$$\begin{tikzcd} L \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & T \arrow[d, "g"] \\ Y \arrow[r, "\Delta_Y"'] & Y \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\circledast"] & Y_S \times_{S'} Y_S \arrow[d, "f_Y \times 1_Y"] \\ & S' \arrow[r, "\Delta_{S'/S}"'] & S' \times_S S' \end{tikzcd}$$

- Etale cover of Y_S:

Y is an alg. space$/S'$ $\Rightarrow$ $\exists\, U \xrightarrow{u} Y$ surj. étale

$(u \in Y(U))$

$g : U \xrightarrow{u} Y \longrightarrow S'$

Then $(g, u) \in Y_S(U) \Leftrightarrow u_S : U \longrightarrow Y_S$ over S

u_S étale cover ?

For $T \to S$ a scheme and an object $(\varepsilon, y) \in Y_S(T)$

$$\begin{tikzcd} U \arrow[r, "u_S"] & Y_S \\ \bullet \arrow[u] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & T \arrow[u, "(\varepsilon, y)"'] \end{tikzcd}$$

is this étale surj of schemes ?

$$\begin{tikzcd}
 & & Y \arrow[from=ddll, "y"] \\
 & & \\
U \arrow[rr, "u_S"] \arrow[uurr] & & Y_S \arrow[uu] \\
 & & \\
\arrow[uu] \arrow[rr] & & T \arrow[uu, "(\varepsilon, y)"'] \arrow[uuuu, bend right=60, "y"']
\end{tikzcd}$$

Hence $U \times_{u_S, Y_S, (\varepsilon, y)} T = U \times_{u, Y, y} T \longrightarrow T$

$\big\downarrow$

étale surjective morphism of sheaves.

(ii) $\mathcal{C} \longrightarrow (AS/_{S'})$

$$\begin{matrix} (X, f) & \longmapsto & T & \longmapsto & X(T) \\ \vdots & & \downarrow & & \\ X & & S' & & \\ \downarrow & & & & \\ S' & & & & \end{matrix}$$

morphisms over S'

check that this defines an alg. space and that this functor is the inverse of the one of (ii)

$\blacksquare$

Rmk:

Take $S = \operatorname{Spec} \mathbb{Z}$

The proposition says that, for every scheme S'

$$(AS_{/S'}) \text{ equivalent to } \mathcal{C}$$

pairs (X, f)

alg space over $\operatorname{Spec} \mathbb{Z}$

$f : X \to S$ Spec $\mathbb{Z}$ - morphism.

So one can first develop the theory of algebraic spaces as an absolute theory, and then obtain the notion of algebraic space over S' any scheme.

(I) Algebraic spaces a sheaf quotients

We give here an alternative definition of alg. space.

Definition (étale equivalence relation)

S a scheme.
R and X S-schemes.

We say that a morphism $R \to X \times_S X$ is an equivalence relation if:

(i) $R \to X \times_S X$ is a monomorphism.

(ii) $\forall\, T \to S$ a scheme, $R(T) \hookrightarrow X(T) \times X(T)$ is an equivalence relation.

(iii) $s, t : R \to X \times_S X \rightrightarrows X$ are étale.

Proposition:

Let F be an algebraic space over S.

Let $X \to F$ be a surjective étale morphism with $X \to S$ a scheme.

Set $R := X \times_F X$. Then
a scheme

(1) $j : R \to X \times_S X$ defines an equivalence relation on X over S.

(2) The diagram

$$R \rightrightarrows X \xrightarrow{\ell} F$$

is a coequalizer in $(Sch/S)_{ét}$.
In particular, F is the étale sheafification of X/R.

<u>Proof</u>:

(1) $T \to S$ a scheme.
$R(T) = \{ (a,b) \in X(T) \times X(T) \mid \ell \circ a = \ell \circ b \}$
which clearly defines an equivalence relation on $X(T)$.

- We want to prove that $R = X \times_F X \to X \times_S X \to X$ is étale.

$$\begin{tikzcd} X \arrow[r] & F \\ R = X \times_F X \arrow[u, "t"] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & X \arrow[u] \end{tikzcd}$$

étale

étale cov

étale

(2) • $X \xrightarrow{l} F$ surjective morphism of sheaves:

$\xi \in F(T)$ with $T \to S$ a scheme.

let $V := T \underset{\xi, F, l}{\times} X \longrightarrow T$

étale surj of schemes

hence $\{V \to T\}$ is a covering in the étale topology.

$$\begin{tikzcd}
T \arrow[r, "\xi"] & F \\
V \arrow[u] \arrow[r] \arrow[ur, "\xi|_V"] \arrow[ur, phantom, very near start, "\lrcorner"] & X \arrow[u, "l"']
\end{tikzcd}$$

$$X(V) \xrightarrow{l} \underset{\substack{\downarrow \\ \xi|_V}}{F(V)}$$

hence $X \to F$ is surjective.

• Given two morphisms $a, b : T \to X$ st $l \circ a = l \circ b$, $\exists\, c : T \to R$ st $a = pr_1 \circ c$ and $b = pr_2 \circ c$, which clear from the definition of R.

Question:

What about the converse?
Given an étale equivalence relation R,
on X over S, then
is a scheme
the étale sheafification $X/_R$ (coequalizer) of

$$T \mapsto X(T)/_R$$

is an algebraic space.

i.e $X/_R \longrightarrow X/_R \times X/_R$ is representable
and $X \to X/_R$ is étale surjective.

Example :

X a scheme $/S$, G discrete gp acting on X

$$f : G \times X \longrightarrow X$$
$$\begin{array}{c} G \times X \\ \wr\wr \\ \coprod_{g \in G} X \end{array}$$
$$(g, x) \longrightarrow f(g, x)$$

The action is said free if the map

$$G \times X \xrightarrow{j} X \times_S X$$
$$(g, x) \longmapsto (x, f(g, x))$$

is a monomorphism.

Then j is an étale equivalence relation.

X/G is then an algebraic space.