Examples of Alg. Spaces.

Reminder:

Def:

S sch, an étale equiv. relation on an S-scheme X is a monomorphism of schemes:

$$R \hookrightarrow X \times_S X$$

such that:

i) For every S-sch. T,

$$R(T) \subset X(T) \times X(T)$$

is an equiv. relation on $X(T)$.

ii) The maps

$$s, t : R \to X$$

induced by the projections from $X \times_S X$ are étale.

We write X/R for the sheaf associated to the presheaf $(Sch/S)^{op} \to Set$, $T \mapsto X(T)/R(T)$.

X/R is an alg. space

Examples:

I) Let X be a scheme, and G a discrete group acting on X, suppose that the action of G is free, that is: the map

$$j : G \times X \to X \times X$$
$$(g, x) \mapsto (x, g \cdot x)$$

is a monomorphism

Let $X/_G$ be the sheaf associated to the presheaf

$$T \in Sch \longmapsto X(T)/_G$$

Then $X/_G$ is the quotient of X by the étale equiv. rel. j
therefore $X/_G$ is an alg. space.

Q: Is $X/_G$ (always) a scheme ?

A: No.

II) Let k be a field of char. 0, and let $\mathbb{Z} \curvearrowright \mathbb{A}^1_k$ by translation. This is a free action, so $X := \mathbb{A}^1_k/_{\mathbb{Z}}$ is an alg. space, but it's not a scheme.

Proof: Suppose X is a scheme:

- X has an étale surjective covering by $\mathbb{A}^1_k$, so X must be smooth and connected.

- For any aff. open $U \subset X$, the sections $\mathcal{O}_X(U)$ must be $\mathbb{Z}$-invariant sections of an open of $\mathbb{A}^1_k$, so rational functions. Therefore they must be constant (since a non cst rational funct. cannot have infinitely many zeros and poles), so $\mathcal{O}_X(U) = k$ for any aff. open $U \subset X$. This implies that

$$X = \operatorname{Spec} k$$

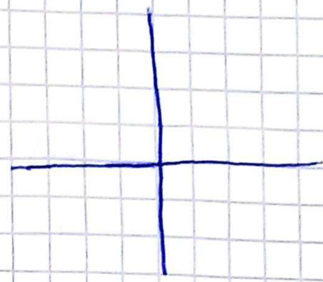
- Remember that $\mathbb{A}^1_k \longrightarrow X$ is an étale covering, this is a contradiction ↯.

So $\mathbb{A}^1_k/\mathbb{Z}$ is not a scheme.

By a similar argument, $\mathbb{A}^1_{\mathbb{C}}/\mathbb{Z}^2$ is an alg. space which is not a scheme (whereas in the analytic top. $\mathbb{C}/\mathbb{Z}^2$ is an elliptic curve!).

III) Let k be a field and

$$U := \mathrm{Spec}\left(k[s,t]/(st)\right)$$



Let $U' \subset U$ be the open obtained by deleting the origin.

Set

$$R := U \sqcup U'$$ and define $\pi_1, \pi_2 : R \to U$

by:

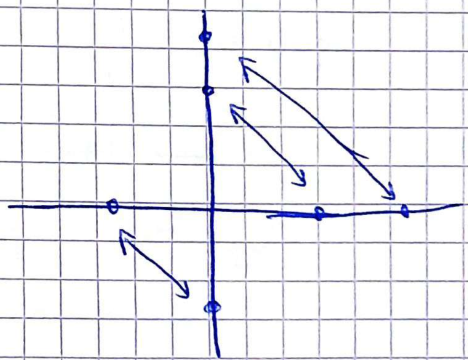
$$\pi_{1|U} = \pi_{2|U} = id_U$$

$\pi_{1|U'} = U' \hookrightarrow U$ the natural inclusion.

$\pi_{2|U'}$: the map which switches the two components.

Then we have an étale equiv. rel.

$$\pi_1 \times \pi_2 : R \hookrightarrow U \times U$$



Let $F = U/R$ be the resulting alg space.

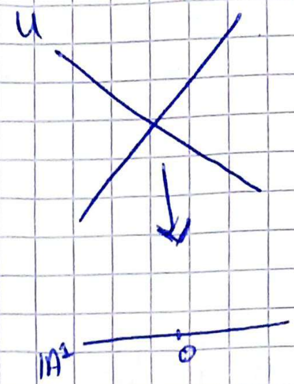
claim: F is not a scheme

Proof: • The map $s+t : U \to \mathbb{A}^1$ satisfies the universal property of the quotient U/R

• So if F were a scheme, ~~we could have an~~ the induced map $F \to \mathbb{A}^1$ must be an isomorphism of schemes, but that's not possible, since the map $U \xrightarrow{s+t} \mathbb{A}^1$ is not étale

$$\begin{tikzcd} U \arrow[r, "\text{non étale}"] \arrow[dr, "\text{étale}"'] & \mathbb{A}^1 \\ & F \arrow[u, dashed, "\text{isom}"'] \end{tikzcd}$$

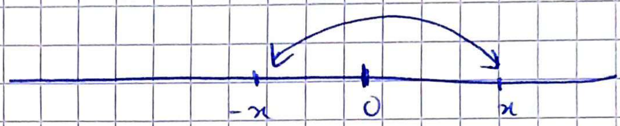
$\lightning$



Similar exple:

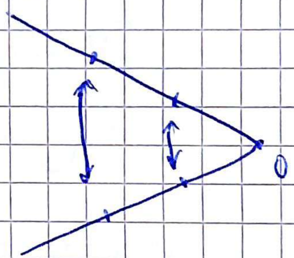
Consider $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\} \curvearrowright \mathbb{A}^1_k$ (char $k \neq 2$)

This is not free. Remove -1 from the stabilizer of 0 to get a free action, call it R.



then $\mathbb{A}^1_k/R$ is not a scheme.

(One way to see it is that $\mathbb{A}^1_k/R$ is isomorphic to the



quotient of 

~~by the relation which iden~~

by the map which switches the two origins and acts as -1 on the rest)

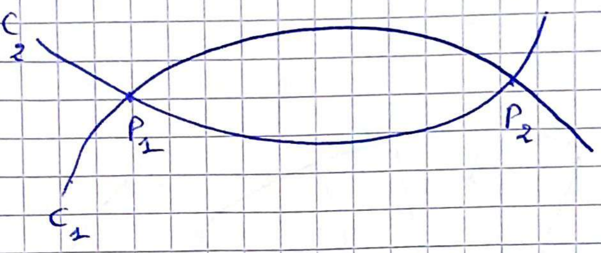
IV) The Example of Hironaka:

Let $\mathbb{P} = \mathbb{P}^3_{\mathbb{C}}$ (w/ coord. Y_0, Y_1, Y_2, Y_3), define curves:

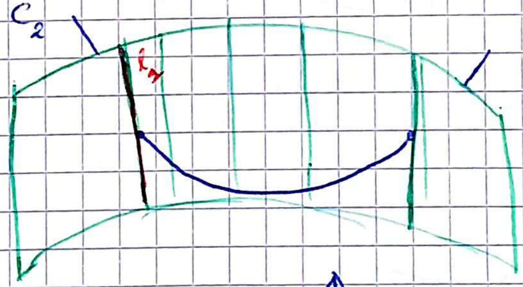
$$C_1 : \quad Y_0Y_1 + Y_1Y_2 + Y_2Y_0 = Y_3 = 0$$

$$C_2 : \quad Y_0Y_1 + Y_1Y_3 + Y_3Y_0 = Y_2 = 0$$

$$C_1 \cap C_2 = \{ P_1 = [1:0:0:0],\ P_2 = [0:1:0:0] \}$$

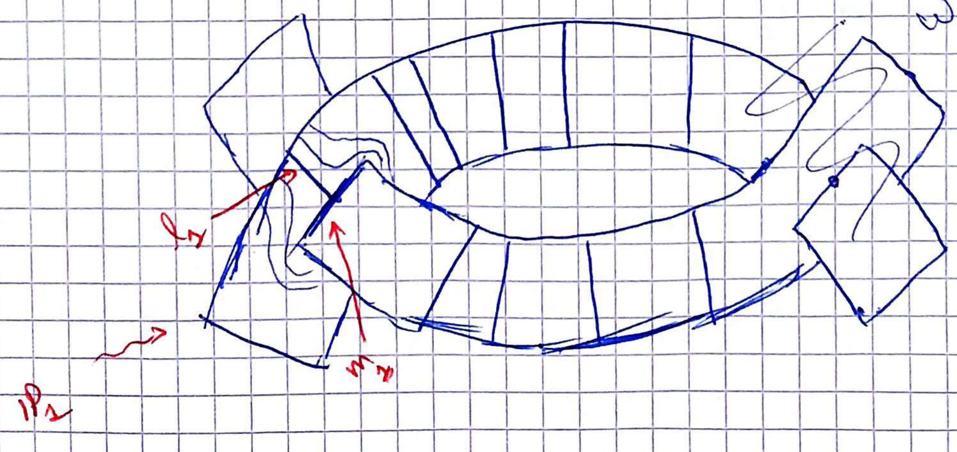


Blow up C_1



Blowup C_2

wrong picture!

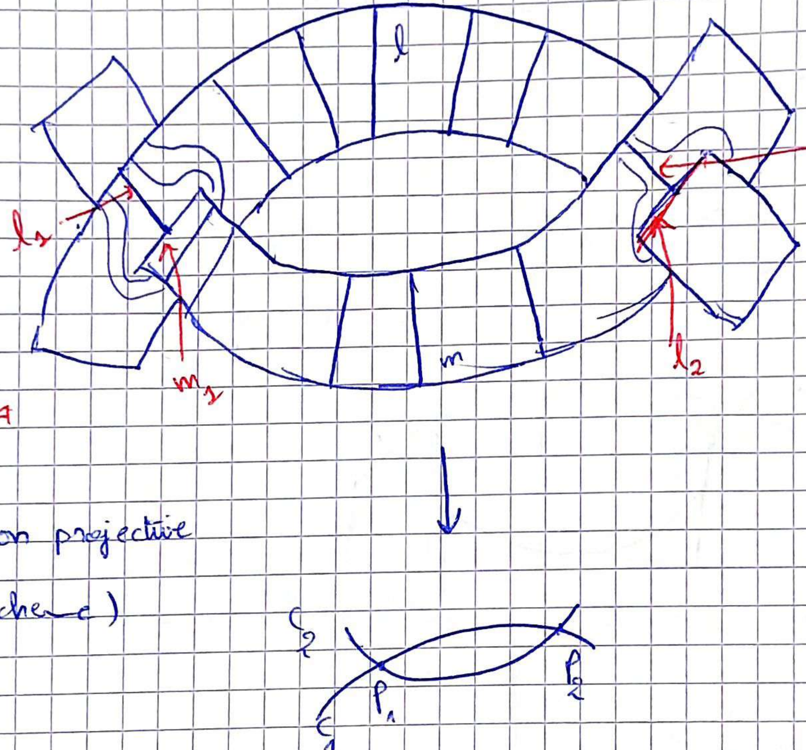


Do the same procedure but start w/ C_2 : we get a scheme $\mathbb{P}_2$.

Let $U_i \subset \mathbb{P}_i$ be the preimage of ~~$\mathbb{P} - \mathbb{P}$~~ $\mathbb{P} - P_{3-i}$

$$\begin{cases} U_1 : \text{preimage of } \mathbb{P} - P_2 \\ U_2 : \quad '' \quad '' \quad \mathbb{P} - P_1 \end{cases}$$

Let Z be the scheme obtained by gluing U_1 and U_2 along their intersection (the preimage of $\mathbb{P} - \{P_1, P_2\}$)



Z

(smooth non projective scheme)

the preimage of P_1 : $l_1 + m_1$

'' '' '' P_2 : $l_2 + m_2$

There is an involution

$$\delta : Z \longrightarrow Z$$

induced by the automorphism of $\mathbb{P}$

$$[Y_0 : Y_1 : Y_2 : Y_3] \mapsto [Y_1 : Y_0 : Y_3 : Y_2]$$

Let $Z' \subset Z$ the open of Z where δ acts freely.

Claim. The quotient $Q := Z'/_{\delta}$ is not a scheme.

Proof: Suppose Q is a scheme.

- Since Z' is smooth and the action of δ is free, Q must be smooth.

- the preimage of P_1 in Z is $l_1 + m_1$; the preimage of P_2 in Z is $l_2 + m_2$, and

$$l_1 + m_2 \sim 0$$

here's why:

By properties of the blowup:

$$\begin{cases} l_1 + m_1 \sim l \\ m_1 \sim m \end{cases} \quad \text{and} \quad \begin{cases} l_2 \sim l \\ l_2 + m_2 \sim m \end{cases}$$

$$\Rightarrow l_1 + m_2 \sim l_1 + m_1 + m_2 + l_2 - m_1 - l_2 \sim 0$$

- ~~Choose a general surface~~

- Since δ interchanges l_1 and m_2, their image in Q is an irred curve $t \subset Q$.

- Choose a general $W \subset Q$ which meets but doesn't contain t, let $\overline{W} \subset Z$ be the closure of its preimage in Z.

Then $\overline{W}$ is a surf. which meets but doesn't contain

$l_1 + m_2$, so

$$\overline{W} \cdot (l_1 + m_2) > 0$$

this is a contradiction! $(l_1 + m_2 \sim 0)$

So Q is not a scheme.

Q: When is the quotient of a sch. by an ét. equiv relation a scheme?

A: We don't know... But there are a few special cases.

~~Theo.~~

IV) ~~Exy~~ Let U be an affine scheme and $R \rightrightarrows U$ a finite étale equiv rel. (So R is affine too!)

Write $U = \operatorname{Spec} A_0$, $R = \operatorname{Spec} A_1$, $A_0 \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} A_1$ the morphisms corresponding to $R \rightrightarrows U$.

Let $B = \operatorname{Eq}(A_0 \overset{\delta_0}{\underset{\delta_1}{\rightrightarrows}} A_1)$.

Theorem: ~~U/R~~ The alg. space U/R is isom. to the affine scheme $\operatorname{Spec} B$.

Cor: • Let $U = \operatorname{Spec} A$ and G finite group acting freely on U, then $U/G \simeq \operatorname{Spec} A^G$.

• If G is a finite group acting freely on a sch. X, if every orbit of ~~X~~ is this action is contained in an aff. open, then X/G is a scheme.

This is of course not the only way to obtain a quotient that is a scheme, look at the action $\mathbb{C}^* \curvearrowright \mathbb{A}^{n+1}_{\mathbb{C}} \setminus \{0\}$, (the action of scaling), the group is infinite but the quotient is $\mathbb{P}^n_{\mathbb{C}}$.