Talk w/ Amira: More about algebraic spaces:

§ 5.4. Basic properties of algebraic spaces

Fix a base scheme S.

Def: (for alg. spaces) Let P = property of schemes, stable in étale topology and let X be an alg. space. Then:

X has property $P \overset{def}{\iff} \exists$ étale surjection $U \to X$ where U is scheme with property P.

Recall: P is a property of schemes stable in étale topology if: X has property $P \iff \exists$ étale cover $\{X_i \to X\}_{i \in I}$ s.th. X_i have prop. P.

e.g.: P = locally noeth., reduced, regular, purely n-dim'l, normal etc.

$\rightsquigarrow$ for all „local properties" ✓

① **Def:** (Morphisms) Let $f: X \to Y$ be a morphism of alg. spaces, repres. by schemes and P = prop. of morph. of schemes stable in ét. topology, (*) then f has a property P if there is an étale cover $V \to Y$ (scheme) and the projection $\underbrace{V \times_Y X}_{\text{these are schemes ✓}} \to V$ has property P.

e.g.: P = proper, dominant, quasi-compact, embedding, open / closed embedding

(*) Recall, that this means: (Olsson def. 5.1.3)

1) P is stable under base change.

2) For all $f: X \to Y$ in (Sch/S) and every étale covering $\{Y_i \to Y\}$, we have:

$$f \text{ has } P \iff f_i : X \times_Y Y_i \to Y_i \text{ have } P \ \forall i.$$

Rmk1: The two def's above are equivalent to the (apriori stronger) statement that X or $f: X \to Y$ has P iff for every étale cover the base change has P. (Details see: Alper's notes, ch. 12)

⇝ use diagonal morphism to define various seperation properties of morphisms:
+ don't need to require that $f: X \to Y$ repres. by schemes.

Def: A morphism $f: X \to Y$ of alg. spaces $/S$ is quasi-separated (resp. locally seperated, seperated) if the diagonal morphism:

$$\Delta_{X/Y} : X \longrightarrow X \times_Y X$$

is quasi-compact (resp. an embedding, a closed embedding)

An alg space X/S is quasi-seperated (resp. locally sep, sep.) if the structure morphism $X \to S$ is quasi-compact (resp. loc. sep, sep) subtlety: Note that $\Delta_{X/Y}: X \to X \times_Y X$ is representable by schemes, since $\Delta_X: X \to X \times_S X$ is. More precisely:

Let T be a scheme and $T \to X \times_Y X$ a morphism, since $X \times_Y X \hookrightarrow X \times_S X$ is a mono, we obtain an isomorphism: $X \times_{\Delta_{X/Y}, X \times_Y X} T \cong \underbrace{X \times_{\Delta_X, X \times_S X}}_{\text{this is a scheme bc. } \Delta_X \text{ is repres. ✓}} T,$

Example (Algebraic space that is not separated)

Let $S = \mathrm{Spec}(\mathbb{Q})$ and let $X = \mathbb{A}^1_S / \mathbb{Z}$ where $\mathbb{Z} \curvearrowright \mathbb{A}^1_S$ by $n * x := x + n$ (free ✓)

this action $\leftrightarrow$ equivalence relation on $\mathbb{A}^1_S$:

= union over all $n \in \mathbb{Z}$ over the morphisms:

$$\mathbb{A}^1_S \hookrightarrow \mathbb{A}^1_S \times_S \mathbb{A}^1_S \qquad x \mapsto (x, x+n)$$

Observe, that we have a cartesian diagram:

$$\begin{tikzcd} \coprod_{n \in \mathbb{Z}} \mathbb{A}^1_S \arrow[r, "\text{not q.c.}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{A}^1_S \times_S \mathbb{A}^1_S \arrow[d, "\text{étale}"] \\ X \arrow[r, "\Delta_X"] & X \times_S X \end{tikzcd}$$

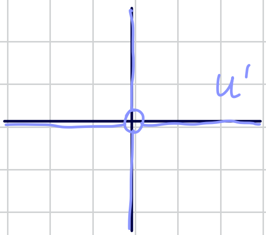
By def. + Rmk 1 $\Rightarrow \Delta_X$ not quasi-compact

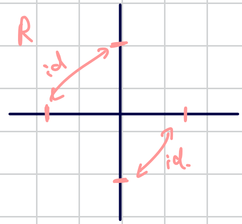
$\Rightarrow X$ not quasi-separated.

Example from earlier: $\mathcal{U} = k[s,t]/(st)$

$\cup$

$\mathcal{U}' = \mathcal{U}\setminus\{0\}$



$F =$ 

$$\mathcal{U} \amalg \mathcal{U}' = R \hookrightarrow \mathcal{U}\times\mathcal{U} \quad \text{étale eq. relation.}$$

Claim: F is locally sep., but not separated.

Consider the cartesian diagram:

embedding, but not closed (image $= \mathcal{U}'\times\mathcal{U}'$ open)

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{U}\times\mathcal{U} \arrow[d, "\pi"', "\text{ét}"] \\ F \arrow[r, "\Delta_F"] & F\times_{\mathcal{U}} F \end{tikzcd}$$

Per def., we have Δ_F is an embedding, but not closed.

$\Rightarrow$ F is locally separated, but not separated.

2

Def: Let P = prop. of morph. of schemes, stable + local on domain in étale topology.

Let $f: X \to Y$ morph. of alg. spaces.

Then f has P

$\Leftrightarrow$ $\exists$ étale covers $v: V \to Y$ and $u: U \to X$, s.th:

$$\underbrace{U \times_Y V}_{\text{scheme!}} \to V$$

has property P.

recall Lemma 5.19.

X alg space $\Rightarrow$ any $T \to X$ is representable by schemes (bc. Δ_X is repres.)

scheme

Recall: stable on domain: $\forall f: X \to Y$ in (Sch/S) and any étale cover $\{ X_i \xrightarrow{x_i} X \}$:

$$f \text{ has } P \Leftrightarrow f \circ x_i \text{ has } P.$$

Applicable for: étale, flat, smooth, surjective, locally of finite type etc.

Remark / Exercise: If $f: X \to Y$ is repres. by schemes and P = stable + loc. on domain then this def + definition in the beginning coincide.

Some global properties involving quasi-compactness

⇝ Note: quasi-compactness is not a morphism of schemes stable in étale topology, so it hasn't been def yet.

Def: • X alg space $/S$ quasi-compact $\Leftrightarrow$ $\exists$ étale cover $U \to X$ with U quasi-compact.

• X noeth $\overset{\text{def}}{\Leftrightarrow}$ locally noeth + quasi-compact.
(stable with resp. to étale top)

• Let $f: X \to Y$ be a morphism of alg. spaces $/S$. Then:

(3) f quasi-compact $\overset{\text{def}}{\Leftrightarrow}$ For any $Y' \to Y$ (q.c.) the alg. space $X' = X \times_Y Y'$ is quasi-compact.

• $f: X \to Y$ is of finite type $\Leftrightarrow$ f q.c. and locally of finite type (sense of Def (2))

Rmk: If $f: X \to Y$ morph. of alg spaces = repres. by schemes

$\Rightarrow$ Def (3) = Def (1)

§ 5.5: Algebraic spaces are fppf sheaves

Thm: Let X be an alg. space $/S$ with quasi-compact diagonal.
Then X is a sheaf wrt. the fppf-topology on (Sch/S).

Rmk:

1) Some authors include quasi-compactness of diagonal into their def of alg space ($\Rightarrow \Delta_X$ is quasi-affine) and this makes descent arguments easier

note

+ Stacks proj. does not require that Δ_X is quasi-proj.

2) One can show that defining alg spaces as a sheaf on $(Sch/S)_{fppf}$ is equivalent to the definition of alg. spaces as sheaves on $(Sch/S)_{Et.}$ (See Stacks Project: [076M]).