

# Examples of algebraic stacks + properties of morphisms

①

Example:

Let  $X$  be an algebraic space, and let  $G/S$  be a smooth group scheme acting on  $X$ . Define  $[X/G]$  to be the stack whose objects are  $(T, \mathcal{P}, \pi)$  w/

- $T$  an  $S$ -scheme.
- $\mathcal{P}$  is a  $G_T := G \times_S T$ -torsor on  $T$ .
- $\pi: \mathcal{P} \rightarrow X \times_S T$  a  $G_T$ -equivariant morphism of sheaves on  $(\text{Sch}/T)$ .

A morphism  $(T', \mathcal{P}', \pi') \rightarrow (T, \mathcal{P}, \pi)$  is a pair  $(f, \tilde{f})$  w/  $f: T' \rightarrow T$  an  $S$ -morphism of schemes, and  $\tilde{f}: \mathcal{P}' \xrightarrow{\sim} f^* \mathcal{P}$  an isom of  $G_T$ -torsors on  $(\text{Sch}/T')$  st

$$\begin{array}{ccc}
 \mathcal{P}' & \xrightarrow{\tilde{f}} & f^* \mathcal{P} \\
 \searrow \pi' & \cong & \swarrow f^* \pi \\
 & X \times_S T &
 \end{array}$$

Claim:  $[X/G]$  is an algebraic stack.

Proof: •  $[X/G]$  is a stack.

• Representability of the diagonal:

Let  $T$  be an  $S$ -scheme and  $(\mathcal{P}_i, \pi_i)_{i=1,2}$  be two objects of  $[X/G]$  over  $T$ .

Using what Céline said earlier, we need to prove that the sheaf  $I := \underline{\text{Isom}}((\mathcal{P}_1, \pi_1), (\mathcal{P}_2, \pi_2))$  is an algebraic space.

$$I \cdot \begin{pmatrix} T' \\ \downarrow \\ T \end{pmatrix} \mapsto \left\{ \begin{array}{ccc} \beta: \mathcal{P}_1|_{T'} & \xrightarrow{\sim} & \mathcal{P}_2|_{T'} \\ & \searrow \beta & \swarrow \beta \\ & X \times_S T' & \end{array} \right\}$$

We may by exercise 5.6 replace  $T$  by an étale covering, so we may assume  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are trivial torsors (so  $G \times T$ ).

Then  $\pi_i: G_T \rightarrow X_T$ .

What is  $I$  in this case?

$$I \cdot \begin{pmatrix} T' \\ \downarrow \\ T \end{pmatrix} \mapsto \left\{ g \in G(T') \text{ st. } \begin{array}{ccc} G_{T'} & \xrightarrow{m_g} & G_{T'} \\ \pi_1' \searrow & \beta & \swarrow \pi_2' \\ & X_{T'} & \end{array} \right\}$$

right multiplication by  $g$

so  $I(T') = \{ g \in G(T') \text{ st. } \pi_1'(e) = \pi_2'(g) \}$ , thus  $I$  is identified w/ the fiber product of

$$\begin{array}{ccc} & G_T & \\ & \downarrow \pi_1'(e) \times \pi_2' & \\ X_T & \xrightarrow{\Delta} & X_T \times X_T \end{array}$$

So  $I$  is a scheme in the case where  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are trivial, therefore it's an alg. space in general.

• Smooth surjective covering by a scheme:

(3)

Let  $q: X \rightarrow [X/G]$  be the map defined by  $(G_X, \beta)$  where  $G_X$  is the trivial group over  $X$ , and  $\beta: G \times X \rightarrow X$  is the action map.

$$\left( \text{so } q: \begin{array}{c} T \\ \downarrow \\ X \end{array} \mapsto (T, G_T, \beta: G \times_T X \rightarrow X \times_T T) \right)$$

$q$  is a smooth surjection:

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & T \\ \downarrow \Gamma & & \downarrow (\beta, \pi) \\ X & \longrightarrow & [X/G] \\ & & (G_X, \beta) \end{array}$$

Etale locally,  $\beta$  is a smooth group scheme over  $T$ ,

so  $q$  is smooth surj.

□

Examples / Def:

• If  $G/S$  is a smooth group scheme, the classifying stack of  $G$ , denoted  $BG$  or  $B_S G$ , is the stack  $[S/G]$  where  $G$  acts trivially on  $S$ .

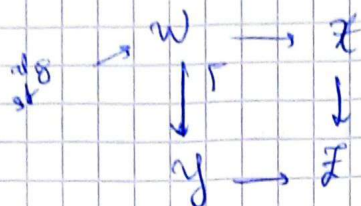
• Let  $\alpha_0, \dots, \alpha_n \in \mathbb{Z}$ , we define the weighted projective stack  $\tilde{\mathbb{P}}(\alpha_0, \dots, \alpha_n) := [ \mathbb{A}^{n+1} \setminus \{0\} / G_m ]$ , where the action of  $G_m$  is given by:

$$u \cdot (x_0, \dots, x_n) = (u^{\alpha_0} x_0, \dots, u^{\alpha_n} x_n)$$

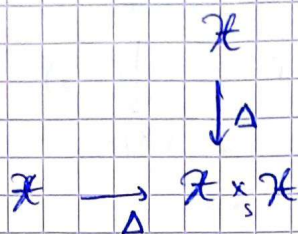
# The inertia stack:

(4)

Proposition: The fibered product of algebraic  $S$ -stacks is an alg.  $S$ -stack.



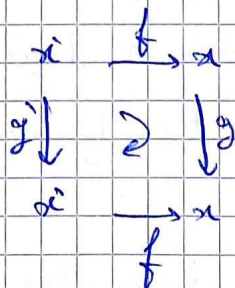
Def: Let  $X/S$  be an alg. stack. The inertia stack of  $X$ , denoted  $I_X$ , is the fiber product of the diagram:



We can describe it explicitly in the following way:

Obj: pairs  $(x, g)$  w/  $x \in X(T)$  and  $g$  is an autom. of  $x$  in  $X(T)$ .

A morphism  $(x', g') \rightarrow (x, g)$  is a ~~map~~ morphism  $f: x' \rightarrow x$  st.



$$\begin{array}{l}
 p: I_X \rightarrow X \\
 (x, g) \mapsto x
 \end{array}$$

Intuitively:  $\mathcal{I}_X$  "measures how close  $X$  is to being a sheaf".

Examples:

• Take  $X = B_S G = [S/G]$  where  $G/S$  is a smooth group scheme, take  $S = \text{Spec } k$ ,  $k$  field of char 0.

Then  $X$  has a single point (the trivial torsor over a pt).

So  $\mathcal{I}_X = [G/G]$  where  $G \curvearrowright G$  by conjugation.

To see this, let's look at objects of  $\mathcal{I}_X$  over  $\text{Spec } k = P$

$\text{Obj} = \{ (p, g), g \in G \}$

$$\begin{array}{ccc}
 (p, g) \rightarrow (p, g') & = & P \xrightarrow{R} P \quad (g' = hgh^{-1}) \\
 & & \downarrow g \quad \downarrow g' \\
 & & P \xrightarrow{R} P
 \end{array}$$

From this we can identify  $\mathcal{I}_X$  with  $[G/G]$

•  $X = [A^2 / (\mathbb{Z}/2\mathbb{Z})]$  where  $\mathbb{Z}/2\mathbb{Z}$  acts by multiplication by  $-1$ .

The inertia stack  $\mathcal{I}_X$  in this case is  $[A^2 / (\mathbb{Z}/2\mathbb{Z})] \sqcup B(\mathbb{Z}/2\mathbb{Z})$

corresponds to the autom  $-1$  on the origin

•  $X = [A^3 / (\mathbb{Z}/2\mathbb{Z})]$  where  $\mathbb{Z}/2\mathbb{Z}$  acts via the following involution:

$(a, b, c) \mapsto (a, -b, -c)$

then  $\mathcal{I}_X = [A^3 / \mathbb{Z}/2\mathbb{Z}] \sqcup B_{A^1}(\mathbb{Z}/2\mathbb{Z})$

## Properties of morphisms of algebraic stacks:

(6)

Def: Let  $P$  be a property of  $S$ -schemes which is stable in the smooth topology. We say that an algebraic stack  $\mathcal{X}/S$  has  $P$  if there exists a smooth surj. morphism  $\pi: X \rightarrow \mathcal{X}$  w/  $X$  scheme having  $P$ .

Examples of  $P$ : locally noetherian, regular, locally of finite type over  $S$ , loc. of f.p. over  $S$ .

Lemma: Let  $P$  be a property of sch. stable w.r.t smooth top., let  $\mathcal{X}/S$  be an alg. stack having  $P$ . Then for any smooth morph.  $\gamma: Y \rightarrow \mathcal{X}$  w/  $Y$  alg. space,  $Y$  has  $P$ .

Proof: Let  $\pi: X \rightarrow \mathcal{X}$  smooth surj. w/  $X$  having  $P$ , we have

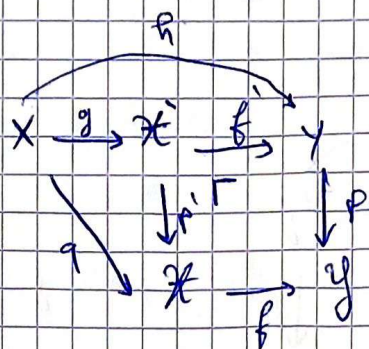
$$\begin{array}{ccc} Y \times_{\mathcal{X}} X & \xrightarrow{b} & X \\ \downarrow a & & \downarrow \pi \\ Y & \xrightarrow{\gamma} & \mathcal{X} \end{array}$$

where  $a$  is smooth surj., and  $b$  is smooth.

Since  $X$  has  $P$ ,  $Y \times_{\mathcal{X}} X$  has  $P$ , so  $Y$  has  $P$ .  $\square$

Now to properties of morphisms:

Def: Let  $f: \mathcal{X} \rightarrow \mathcal{Y}$  be a morph. of alg stacks. A chart for  $f$  is the commutative diagram:



where  $X$  and  $Y$  are alg spaces,  $g$  and  $p$  smooth surjective.

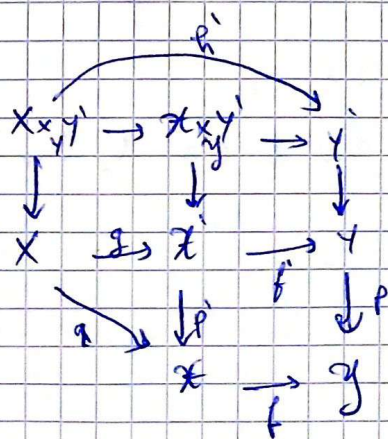
Def: Let  $P$  be a property of morphisms of schemes which is stable and local on domain wrt smooth top. We say that a morphism  $f: X \rightarrow Y$  has  $P$  if there exists a chart for  $f$  by schemes such that  $h$  has  $P$ .

Examples of  $P$ : smooth, locally of finite presentation, surjective.

Prop: Let  $P$  as before,  $f: X \rightarrow Y$  has  $P$  iff for every chart of  $f$ ,  $h$  has  $P$ .

Proof:

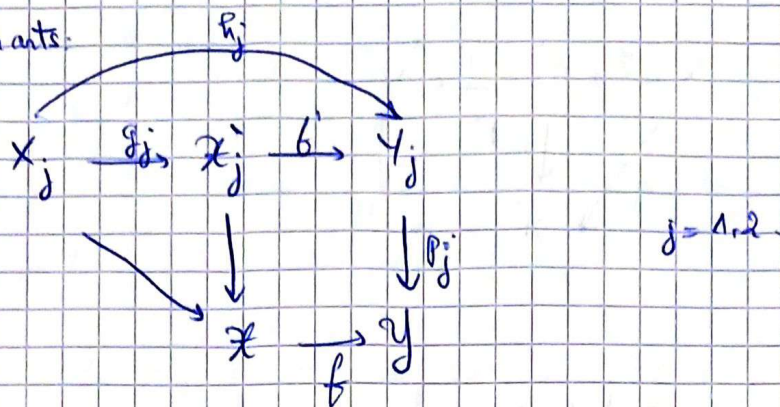
• Fix a chart, and let  $Y' \rightarrow Y$  be a smooth surj. morph. of alg spaces, we get the diag.:



the outside diagram is also a chart for  $f$ .

Since  $X \times_Y Y' \rightarrow X$  is smooth surj.,  $P$  has  $P \Leftrightarrow \tilde{P}$  has  $P$ . ⑧

• Fix two charts:



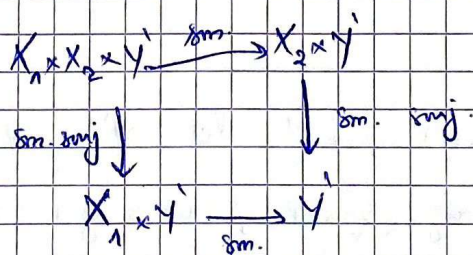
Goal:  $P_1$  has  $P \Leftrightarrow P_2$  has  $P$ .

Set  $Y' = Y_1 \times_Y Y_2$ . Applying the previous discussion to the smooth surj. maps  $pr_j: Y' \rightarrow Y_j$

we get  $P_1$  has  $P \Leftrightarrow X_1 \times Y' \rightarrow Y'$  has  $P$

$P_2$  has  $P \Leftrightarrow X_2 \times Y' \rightarrow Y'$  has  $P$

So we may assume  $Y_1 = Y_2$  and  $P_1 = P_2$ , so



we see that

$P_1$  has  $P \Leftrightarrow P_2$  has  $P$ .

□



(9)

Def: Let  $P$  be a property of morphisms of algebraic spaces which is stable wrt the smooth top (for alg. sp.). We say that a representable morphism of alg. stacks  $f: \mathcal{X} \rightarrow \mathcal{Y}$  has  $P$  if for every  $V \rightarrow \mathcal{Y}$  w/  $V$  alg. sp., the morph.  $\mathcal{X} \times V \rightarrow V$  has  $P$ .

Exmples of  $P$ : étale, smooth of relative dim  $d$ , separated, proper, affine, finite, unramified, a (closed/open) embedding.

In particular, if  $f: \mathcal{X} \rightarrow \mathcal{Y}$  a morph. of alg. st / S, then the diagonal

$$\Delta_{\mathcal{X}/\mathcal{Y}}: \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \quad \text{is representable,}$$

~~and~~

Def:  $f: \mathcal{X} \rightarrow \mathcal{Y}$  morph. of alg. st.

$$\Delta_{\mathcal{X}/\mathcal{Y}}: \mathcal{X} \rightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

We say that:

- $f$  is quasi-separated if  $\Delta_{\mathcal{X}/\mathcal{Y}}$  is q. cpct and q. separated
- $f$  is separated if  $\Delta_{\mathcal{X}/\mathcal{Y}}$  is proper.
- (If  $\mathcal{Y} = S$  and  $f$  the structure morph., we say  $\mathcal{X}$  is q. sep / separated).