# Examples of algebraic stacks + properties of morphisms

Example:

Let $X$ be an algebraic space, and let $G/S$ be a smooth group scheme acting on $X$. Define $[X/G]$ to be the stack whose objects are $(T, P, \pi)$ w/

- $T$ an $S$-scheme.
- $P$ is a $G_T := G \times_S T$ - torsor on $T$.
- $\pi : P \longrightarrow X \times_S T$ a $G_T$ - equivariant morphism of sheaves on $(Sch/_T)$.

A morphism $(T', P', \pi') \longrightarrow (T, P, \pi)$ is a pair $(f, \tilde{f})$ w/ $f : T' \longrightarrow T$ an $S$-mph. of schemes, and $\tilde{f} : P' \xrightarrow{\sim} f^*P$ an isom. of $G_{T'}$-torsors on $(Sch/_{T'})$ st.

$$\begin{tikzcd} P' \arrow[rr, "\tilde{f}"] \arrow[dr, "\pi'"'] & \circlearrowleft & f^*P \arrow[dl, "f^*\pi"] \\ & X \times_S T' & \end{tikzcd}$$

Claim: $[X/G]$ is an algebraic stack.

Proof:
- $[X/G]$ is a stack.
- Representability of the diagonal:

Let $T$ be an $S$-scheme and $(P_i, \pi_i)_{i=1,2}$ be two objects of $[X/G]$ over $T$.

Using what Céline said earlier, we need to prove that the sheaf $I := \underline{Isom}((P_1, \pi_1), (P_2, \pi_2))$ is an algebraic space.

$$I : \begin{pmatrix} T' \\ \downarrow \\ T \end{pmatrix} \longmapsto \left\{ \begin{tikzcd} \varphi : P_1|_{T'} \arrow[rr, "\sim"] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & P_2|_{T'} \arrow[dl] \\ & X \times_S T' & \end{tikzcd} \right\}$$

We may by exercise 5.G replace $T$ by an étale covering, so we may assume $P_1$ and $P_2$ are trivial torsors (so $G \times T$).

Then $\pi_i : G_T \longrightarrow X_T$.

What is $I$ in this case?

right multiplication by $g$

$$I : \begin{pmatrix} T' \\ \downarrow \\ T \end{pmatrix} \longmapsto \left\{ g \in G(T') \text{ st. } \begin{tikzcd} G_{T'} \arrow[rr, "m_g"] \arrow[dr, "\pi_1'"'] & \arrow[d, phantom, "\circlearrowleft"] & G_{T'} \arrow[dl, "\pi_2'"] \\ & X_{T'} & \end{tikzcd} \right\}$$

~~neutral~~ identity section

so $I(T') = \{ g \in G(T') \text{ st } \pi_1'(e) = \pi_2'(g) \}$, thus $I$ is identified w/ the fiber product of

$$\begin{tikzcd} & G_T \arrow[d, "\pi_1(e) \times \pi_2"] \\ X_T \arrow[r, "\Delta"'] & X_T \times_T X_T \end{tikzcd}$$

So $I$ is a scheme ~~if~~ in the case where $P_1$ and $P_2$ are trivial, therefore it's an alg. space in general.

- Smooth surjective covering by a scheme:

Let $q : X \longrightarrow [X/G]$ be the map defined by $(G_X, \rho)$ where $G_X$ is the trivial torsor over $X$, and $\rho : G \times X \to X$ is the action map.

$$\left( \text{so } q : \begin{array}{c} T \\ \downarrow \\ X \end{array} \longmapsto (T, G_T, \rho : G \times_S T \to X \times_S T) \right)$$

$q$ is a smooth surjection:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "r"'] & T \arrow[d, "(\rho, \pi)"] \\ X \arrow[r, "(G_X, \rho)"'] & {[X/G]} \end{tikzcd}$$

Etale locally, $\rho$ is a smooth group scheme over $T$, so $q$ is smooth surj.

$\square$

Examples / Def:

- If $G/S$ is a smooth group scheme, the classifying stack of $G$, denoted $BG$ or $B_S G$, is the stack $[S/G]$ where $G$ acts trivially on $S$.

- Let $\alpha_0, \dots, \alpha_n \in \mathbb{Z}$, we define the weigthed projective stack $\mathcal{P}(\alpha_0, \dots, \alpha_n) := [\mathbb{A}^{n+1} \setminus \{0\} / \mathbb{G}_m]$, where the action of $\mathbb{G}_m$ is given by:

$$u \cdot (x_0, \dots, x_n) = (u^{\alpha_0} x_0, \dots, u^{\alpha_n} x_n)$$

# The inertia stack:

**Proposition**: The fibered product of algebraic $S$-stacks is an alg. $S$-stack.

$$\begin{tikzcd} \text{alg st} \arrow[r] & \mathcal{W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{X} \arrow[d] \\ & \mathcal{Y} \arrow[r] & \mathcal{Z} \end{tikzcd}$$

**Def**: Let $\mathcal{X}/S$ be an alg stack. The inertia stack of $\mathcal{X}$, denoted $I_{\mathcal{X}}$, is the fiber product of the diagram:

$$\begin{tikzcd} & \mathcal{X} \arrow[d, "\Delta"] \\ \mathcal{X} \arrow[r, "\Delta"'] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

We can describe it explicitely in the following way:

Obj: pairs $(x, g)$ w/ $x \in \mathcal{X}(T)$ and $g$ is an autom. of $x$ in $\mathcal{X}(T)$.

A morphism $(x', g') \to (x, g)$ is a ~~par~~ morphism $f : x' \to x$ st.

$$\begin{tikzcd} x' \arrow[r, "f"] \arrow[d, "g'"'] \arrow[dr, phantom, "\circlearrowleft"] & x \arrow[d, "g"] \\ x' \arrow[r, "f"'] & x \end{tikzcd}$$

$$p : I_{\mathcal{X}} \to \mathcal{X}$$
$$(x, g) \mapsto x$$

Intuitively: $\mathcal{I}_{\mathcal{X}}$ "measures how close $\mathcal{X}$ is to being a sheaf".

Examples:

- Take $\mathcal{X} = B_S G = [S/G]$ where $G/S$ is a smooth group scheme, take $S = \operatorname{Spec} k$, $k$ field of char $0$.

  Then $\mathcal{X}$ has a single point (the trivial torsor over a pt).

  So $\mathcal{I}_{\mathcal{X}} = [G/G]$ where $G \curvearrowright G$ by conjugation.

  To see this, let's look at objects of $\mathcal{I}_{\mathcal{X}}$ over $\operatorname{Spec} k = p$

  $$\mathrm{Obj} = \{(p, g),\ g \in G\}$$

  $$(p,g) \longrightarrow (p,g') \;=\; \begin{tikzcd} p \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, "\circlearrowleft"] & p \arrow[d, "g'"] \\ p \arrow[r, "h"'] & p \end{tikzcd} \qquad (g' = hgh^{-1})$$

  From this we can identify $\mathcal{I}_{\mathcal{X}}$ with $[G/G]$

- $\mathcal{X} = [\mathbb{A}^1/(\mathbb{Z}/2\mathbb{Z})]$ where $\mathbb{Z}/2\mathbb{Z}$ acts by multiplication by $-1$.

  The inertia stack $\mathcal{I}_{\mathcal{X}}$ in this case is $[\mathbb{A}^1/(\mathbb{Z}/2\mathbb{Z})] \sqcup B(\mathbb{Z}/2\mathbb{Z})$

  corresponds to the autom $-1$ on the origin.

- $\mathcal{X} = [\mathbb{A}^3/(\mathbb{Z}/2\mathbb{Z})]$ where $\mathbb{Z}/2\mathbb{Z}$ acts via the following involution:

  $$(a,b,c) \mapsto (a,-b,-c)$$

  then $\mathcal{I}_{\mathcal{X}} = [\mathbb{A}^3/_{\mathbb{Z}/2\mathbb{Z}}] \sqcup B_{\mathbb{A}^1}(\mathbb{Z}/2\mathbb{Z})$

# Properties of morphisms of algebraic stacks.

**Def:** Let $P$ be a property of $S$-schemes which is stable in the smooth topology. We say that an algebraic stack $\mathcal{X}/S$ has $P$ if there exists a smooth surj. mph.
$$\pi : X \to \mathcal{X} \quad \text{w/ } X \text{ scheme having } P.$$

**Examples of $P$:** locally noetherian, regular, locally of finite type over $S$, loc. of f.p. over $S$.

**Lemma:** Let $P$ be a property of sch. stable w.r.t smooth top., let $\mathcal{X}/S$ be an alg. stack having $P$. Then for any smooth mph. $y : Y \to \mathcal{X}$ w/ $Y$ alg. space, $Y$ has $P$.

**Proof:** Let $\pi : X \to \mathcal{X}$ smooth surj w/ $X$ having $P$, we have

$$\begin{tikzcd} Y \times_{\mathcal{X}} X \arrow[r, "b"] \arrow[d, "a"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "y"'] & \mathcal{X} \end{tikzcd}$$

where $a$ is smooth surj, and $b$ is smooth.

Since $X$ has $P$, $Y \times_{\mathcal{X}} X$ has $P$, so $Y$ has $P$.

$\square$

Now to properties of morphisms:

**Def:** Let $f : \mathcal{X} \to \mathcal{Y}$ be a mph. of alg stacks. A chart for $f$ is the commutative diagram:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend left, "h"] \arrow[dr, "q"'] & \mathcal{X}' \arrow[r, "f'"] \arrow[d, "p'"] \arrow[dr, phantom, "\ulcorner"] & Y \arrow[d, "p"] \\ & \mathcal{X} \arrow[r, "f"'] & \mathcal{Y} \end{tikzcd}$$

where $X'$ and $Y$ are alg spaces, $g$ and $p$ smooth surjective.

**Def:** Let $P$ be a property of morphisms of schemes which is stable and local on domain wrt smooth top. We say that a mph. $f : \mathcal{X} \to \mathcal{Y}$ has $P$ if there exists a chart for $f$ by schemes such that $h$ has $P$.

**Examples of $P$:** smooth, locally of finite presentation, surjective.

**Prop:** Let $P$ as before, $f : \mathcal{X} \to \mathcal{Y}$ has $P$ iff for every chart of $f$, $h$ has $P$.

**Proof:**

- Fix a chart, and let $Y' \to Y$ be a smooth surj. mph. of alg. spaces, we get the diag.:

$$\begin{tikzcd} X\times_Y Y' \arrow[r] \arrow[d] \arrow[rr, bend left, "h'"] & \mathcal{X}\times_{\mathcal{Y}} Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "g"] \arrow[dr, "q"'] & \mathcal{X}' \arrow[r, "f'"'] \arrow[d, "p'"] & Y \arrow[d, "p"] \\ & \mathcal{X} \arrow[r, "f"'] & \mathcal{Y} \end{tikzcd}$$

the outside diagram is also a chart for $f$.

Since $X \times_{\mathcal{Y}} Y' \to X$ is smooth surj., $h$ has P $\iff$ $h'$ has P.

- Fix two charts:

$$\begin{tikzcd}
X_j \arrow[r, "g_j"] \arrow[rr, bend left, "h_j"] \arrow[dr] & \mathcal{X}_j \arrow[r, "b'"] \arrow[d] & Y_j \arrow[d, "p_j"] \\
& \mathcal{X} \arrow[r, "f"'] & \mathcal{Y}
\end{tikzcd}$$

$j = 1, 2.$

Goal: $h_1$ has P $\iff$ $h_2$ has P.

Set $Y' = Y_1 \times_{\mathcal{Y}} Y_2$. Applying the previous discussion to

the smooth surj. maps $pr_j : Y' \to Y_j$

we get $h_1$ has P $\iff$ $X_1 \times Y' \to Y'$ has P

$h_2$ has P $\iff$ $X_2 \times Y' \to Y'$ has P

So we may assume $Y_1 = Y_2$ and $p_1 = p_2$, so

$$\begin{tikzcd}
X_1 \times X_2 \times Y' \arrow[r, "sm."] \arrow[d, "sm.\ surj"'] & X_2 \times Y' \arrow[d, "sm.\ surj."] \\
X_1 \times Y' \arrow[r, "sm."'] & Y'
\end{tikzcd}$$

we see that

$h_1$ has P $\iff$ $h_2$ has P.

$\square$

<u>Def:</u> Let $P$ be a property of morphisms of algebraic spaces which is stable wrt the smooth top (for alg. sp.). We say that a representable morphism of alg. stacks

$f: \mathcal{X} \to \mathcal{Y}$ has $P$ if for every $Y \to \mathcal{Y}$ w/ $Y$ alg. sp.,

the mph. $\mathcal{X} \times Y \to Y$ has $P$.

<u>Exples of P:</u> étale, smooth of relative dim $d$, separated, proper, affine, finite, unramified, a (closed / open) embedding ...

In particular, if $f: \mathcal{X} \to \mathcal{Y}$ a mph. of alg. st. $/S$, then the diagonal

$$\Delta_{\mathcal{X}/\mathcal{Y}} : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \quad \text{is representable,}$$

~~and~~

<u>Def:</u> $f: \mathcal{X} \to \mathcal{Y}$ mph. of alg. st.

$$\Delta_{\mathcal{X}/\mathcal{Y}} : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$$

We say that:

- $f$ is quasi-separated if $\Delta_{\mathcal{X}/\mathcal{Y}}$ is q. cpct and q. separated
- $f$ is separated if $\Delta_{\mathcal{X}/\mathcal{Y}}$ is proper.
  (If $\mathcal{Y} = S$ and $f$ the structure mph, we say $\mathcal{X}$ is ~~q~~q. sep / separated).