Talk: Algebraic stacks, part I

Fix a base scheme S and we work over the category of S-schemes equipped with the étale topology.

$\rightsquigarrow$ Before starting, let us recall some essential Def / results:

Def (Stack): A category fibered in groupoids $p: F \to C$ is a stack $\overset{\text{Def}}{\Longleftrightarrow}$ For every object $X \in \mathcal{C}$ & cover $\{X_i \to X\}_{i \in I}$, the functor

$$\varepsilon: F(X) \longrightarrow F(\{X_i \to X\}_{i \in I})$$

is an equivalence.

Morphism of stacks = morphisms of fibered cat $/\mathcal{C}$.

Recall: $\Big\{$ • commutes with the str. morphisms:
$\begin{tikzcd} F \arrow[r] \arrow[dr, "g"'] & C \arrow[d, phantom, "\circlearrowleft"] \\ & G \arrow[u] \end{tikzcd}$
• preserves cartesian morphisms.

Def A morphism of stacks $f: \mathcal{X} \to \mathcal{Y}$ is representable if for every scheme U and morphism $y: U \to \mathcal{Y}$, the fibre product

$$\mathcal{X} \times_{\mathcal{Y}, y} U$$

is an algebraic space.

Note: If $f : \mathcal{X} \to \mathcal{Y}$ is representable, then for every $y : \overset{\text{alg space}}{V} \to \mathcal{Y}$

the fibre product $V \times_{\mathcal{Y}} \mathcal{X}$ is an algebraic space.

prf:

- condition that the obj. of $V \times_{y} \mathcal{X}$ have no non-trivial automorphisms can be checked after étale b.c. $\underset{\text{scheme}}{U} \to V$. $\underset{\text{repres.}}{\overset{f}{\Longrightarrow}}$ $U \times_{y} \mathcal{X}$ algebraic space, in ptc. sheaf.

prf: (contraposition)

$$\begin{tikzcd} \mathcal{X} \times U \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d] \\ \underbrace{\mathcal{X} \times V} \arrow[r] & V \end{tikzcd}$$

Stack a priori

→ show it's a sheaf.

- Now apply Ex. 5G: Y/S alg. space, $F \in Sh((Sch/S)_{et})$ and $g : F \to Y$ morphism of sheaves.

If $\exists$ ét. surj. $\overset{\text{scheme}}{U} \to Y$ sth. $F \times_Y U$ is alg space, then F is an alg. space. ⇝ Apply to $\underset{\text{sheaf}}{V \times_y \mathcal{X}} \to \underset{\text{alg space}}{V}$. #

"for (étale) sheaves living over alg space: being an alg space is étale-local property" proof: (explain only orally)

By assumption $\exists$ étale, surj. $u : \overset{\text{sch}}{U} \to Y$ and $F \times_Y U$ alg. Space.

- repres. of Δ_F: Consider the diagram:

$$\begin{tikzcd} F \times_{\Delta_F} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & T \arrow[d] \\ F \arrow[r, "\Delta_F"] & F \times_Y F \end{tikzcd}$$

+s: this is alg. space. T: scheme

but we know that after étale base change $U \overset{\text{ét}}{\twoheadrightarrow} F$:

$\Delta_{F \times U}$ is representable, i.e.

$$\begin{tikzcd} F \times_Y U \times T \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & T \arrow[d] \\ F \times_Y U \arrow[r, "\Delta"] & (F \times_Y U) \times_Y (F \times_Y U) \end{tikzcd}$$

$F \times_Y U \times T$: scheme.

and since alg. spaces are „schemes glued in étale top" #

- $\exists$ W scheme & étale surj. $W \twoheadrightarrow F \times_Y U \overset{id \times u}{\underset{\text{ét}}{\twoheadrightarrow}} F \times_Y Y = F$

+ étale, surjective is stable under base-change #

Def (alg stack) A stack $\mathcal{X}/S$ is an algebraic stack, if the following hold:
(sometimes „Artin stack")

(i) The diagonal $\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times_S \mathcal{X}$ is representable

(ii) There exists a smooth surjective morphism $\pi : X \to \mathcal{X}$ with X a scheme.
↳ note: at this point it's not clear what this should mean, explanation below!

Morphisms of alg stacks = morphisms of fibered categories.
(= m. of cat. fib. in groupoids)

$\Rightarrow$ have groupoid of morphisms $\mathrm{HOM}_S(\mathcal{X}, \mathcal{Y})$.

Rmk: condition (i) $\Rightarrow$ every morphism $t : T \to \mathcal{X}$ (T scheme) is representable.

Check: Let $u : U \to \mathcal{X}$ (U scheme) be another morphism and consider

$$\begin{tikzcd} U \times_{\mathcal{X}} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & T \arrow[d, "t"] \\ U \arrow[r, "u"] & \mathcal{X} \end{tikzcd}$$

, this fibre product is in fact isomorphic to the fibre prod of:

$$\begin{tikzcd} & U \times_S T \arrow[d, "u \times t"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

$\Rightarrow U \times_{\mathcal{X}} T$ alg. space.

Therefore, $X \to \mathcal{X}$ (X scheme, $\mathcal{X}$ stack) smooth, surj $\overset{\text{def}}{\Longleftrightarrow}$

$\forall$ $U \xrightarrow{u} \mathcal{X}$ the base change $X \times_{\mathcal{X},u} U$ is smooth & surjective.

U scheme

$$\begin{tikzcd} X \times_{\mathcal{X},u} U \arrow[d] \\ U \end{tikzcd}$$

both alg spaces

The following lemma is convenient for verifying that a given stack is algebraic:

Lemma: Let $\mathcal{X}/S$ be a stack. Then the diagonal $\Delta_{\mathcal{X}}$ is representable iff for every S-scheme U and objects $u_1, u_2 \in \mathcal{X}(U)$ the sheaf $\underline{\mathrm{Isom}}(u_1,u_2)$ on Sch/U is an alg. space.

note: it's a sheaf because $\mathcal{X}$ is a stack.

Recall: $\underline{\mathrm{Isom}}(u_1,u_2)$ is sheaf def. by:

$$\underline{\mathrm{Isom}}(u_1,u_2)(f : X \to U) := \mathrm{Isom}_{\mathcal{X}(X)}(f^*u_1, f^*u_2).$$

proof: For every S-scheme U and obj. $u_1, u_2 \in \mathcal{X}(U)$ we have a cartesian diagram:

$$\begin{tikzcd} \underline{\mathrm{Isom}}(u_1,u_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U \arrow[d, "u_1 \times u_2"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

$\square$

Rmk: Together with ex. 5.G we obtain the following more general result:

$\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable $\iff$ for every algebraic space X and morphisms $u_1, u_2 : X \to \mathcal{X}$ the sheaf $\underline{\mathrm{Isom}}(u_1, u_2)$ is alg. space.

Prop: Let $\mathcal{X}/S$ be an algebraic stack. Then for any diagram

$$\begin{tikzcd} & X \arrow[d, "x"] \\ Y \arrow[r, "y"] & \mathcal{X} \end{tikzcd}$$

with X, Y alg. spaces, the fibre product $X \times_{\mathcal{X}} Y$ is an alg. space

In ptc: Any morphism $\underset{\text{alg space}}{X} \longrightarrow \mathcal{X}$ is representable.

proof:

Use results above: $X \times_{\mathcal{X}} Y \simeq \underline{\mathrm{Isom}}(pr^* x, pr^* y)$ over $X \times_S Y$

and since $\Delta_{\mathcal{X}}$ is repres., the claim follows from Rmk $\square$

$\Rightarrow$ Now we can replace condition (ii) in the def of an alg. stack by the cond. that $\exists$ smooth surj. morphism $\pi : X \longrightarrow \mathcal{X}$ with X an alg. space.

MAIN EXAMPLE: $[X/G]$ quotient stack of $\begin{cases} X \text{ alg. space} \\ G/S \text{ smooth group schemes} \end{cases}$

$\hookrightarrow$ for def of $[X/G]$ need notion of G-torsor:

$\approx$ „generalization of a principal G-bundle"

$X \to S$ scheme $\qquad$ $G \to S$ fppf group scheme

Def: Let $\mathcal{C}$ be site and μ sheaf of groups on $\mathcal{C}$. A μ-torsor on $\mathcal{C}$ $\overset{\text{def}}{=}$ $\left\{ \begin{array}{l} \text{sheaf } \mathcal{P} \text{ on } \mathcal{C} \ / \ X\text{-scheme } P \\ \text{left action } \mu \overset{\rho}{\circlearrowright} \mathcal{P} \end{array} \right\}$ such that:

$P(X_i) = \mathrm{Hom}(X_i, P)$ $\qquad$ action $G \overset{\sigma}{\circlearrowright} P$, s.th. $P \to T$ G-invariant, fppf.

$\to$ because we have fppf-locally a section., if G smooth $\Rightarrow$ étale locv.

(T1) $\forall X \in \mathcal{C}$, $\exists$ covering $\{X_i \to X\}$ s.th. $\mathcal{P}(X_i) \neq \emptyset$ $\forall i$.

(T2) The map $\mu \times \mathcal{P} \xrightarrow{\sim} \mathcal{P} \times \mathcal{P}$ $\quad (g, p) \mapsto (p, gp)$ is an iso.

$(\Leftrightarrow)$ $\mu(T) \curvearrowright \mathcal{P}(T)$ is simply transitive for any $T \to X$. (if $\neq \emptyset$)

Let $\mathcal{C} = (Sch/S)_{\acute{e}t}$, we call $\mathcal{P}$ trivial if it has a global section. Fix one $p \in \mathcal{P}(S)$ then we have an isomorphism $\mu \to P$, $g \mapsto gp$, identifying the action ρ with the left-translation on μ.

Morphism of μ-torsors $(P,\rho) \to (P',\rho')$ $\overset{\text{def}}{=}$ morph of sheaves $f\colon P \to P'$, compatible with the respective μ-actions, i.e. the following diagram commutes:

$$\begin{tikzcd}
\mu \times P \arrow[r, "\mathrm{id}_P \times f"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mu \times P' \arrow[d] \\
P \arrow[r, "f"] & P'
\end{tikzcd}$$

Examples:

assigning a T-scheme T' to the set of isomorphisms of the principal G-bundles $P \times_T T'$ and $Q \times_T T'$, is representable by a scheme which is also a principal G-bundle over T.

$f^{-1}(x) =$ basis of V-sp. $\pi^{-1}(x)$

Exercise C.2.11 (Principal GL_n-bundles)**.** Let T be a scheme.

(a) If E is a vector bundle over T of rank n, the *frame bundle* $\mathrm{Frame}_T(E)$ is defined as the functor $\underline{\mathrm{Isom}}_T(\mathcal{O}_T^n, E)$ on Sch/T, i.e

$$\mathrm{Frame}_T(E)\colon \mathrm{Sch}/T \to \mathrm{Sets}$$
$$(T' \to T) \mapsto \{\text{trivializations } \alpha\colon \mathcal{O}_{T'}^n \xrightarrow{\sim} f^*E\}.$$

Show that $\mathrm{Frame}_T(E)$ is representable by a scheme and that $\mathrm{Frame}_T(E) \to T$ is a principal GL_n-bundle.

$$\begin{array}{cc} E & \mathrm{Frame}(E) \\ \downarrow \pi & \downarrow f \\ B & B \end{array}$$

(b) If $P \to T$ is a principal GL_n-bundle, then define $P \times^{\mathrm{GL}_n} \mathbb{A}^n := (P \times \mathbb{A}^n)/\mathrm{GL}_n$ where GL_n acts diagonally via its given action on P and the standard action on $\mathbb{A}^n$. (The action is free and the quotient $(P \times \mathbb{A}^n)/\mathrm{GL}_n$ can be interpreted as the sheafification of the quotient presheaf $\mathrm{Sch}/T \to \mathrm{Sets}$ taking $T \mapsto (P \times \mathbb{A}^n)(T)/\mathrm{GL}_n(T)$ in the big Zariski (or big étale) topology or equivalently as the algebraic space quotient (Corollary 4.4.11)). Show that $(P \times \mathbb{A}^n)/\mathrm{GL}_n$ is representable by scheme and is the total space of a vector bundle over T. *Hint: Use Effective Descent for Principal G-bundles (C.2.5).*

$E \cong \mathcal{O}^n \Rightarrow \mathrm{Frame}(E) = \mathrm{GL}_n$

(c) Conclude that

$$\{\text{vector bundles over } T\} \to \{\text{principal } \mathrm{GL}_n\text{-bundles over } T\}$$
$$E \mapsto \mathrm{Frame}_T(E)$$
$$(P \times \mathbb{A}^n)/\mathrm{GL}_n \leftarrow P$$

defines an equivalence of categories between the *groupoids* of vector bundles over T and principal GL_n-bundles over T.

Exercise C.2.7. If T is a scheme, show that there there is an equivalence of categories

$$\{\text{line bundles on } T\} \xrightarrow{\sim} \{\text{principal } \mathbb{G}_m\text{-bundles on } T\}$$
$$L \mapsto \mathbb{A}(L) \setminus T$$

between the *groupoids* of line bundles on T (where the only morphisms allowed are isomorphisms) and $\mathbb{G}_m$-torsors on T. If L is a line bundle (i.e. invertible $\mathcal{O}_T$-module), then $\mathbb{A}(L)$ denotes the total space $\operatorname{Spec} \operatorname{Sym}^* L^\vee$ of L and $T \subset \mathbb{A}(L)$ denotes the image of the zero section $T \to \mathbb{A}(L)$.