The Étale Fundamental Group

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1 Introduction

When studying geometric objects a typical invariant to consider is the amount of 'holes' it has. For topological spaces this is done using the homotopy groups. In particular, we are interested in the fundamental group. The fundamental group 'detects holes' by studying maps from the circle into the space up to a deformation called homotopy. This homotopy relies on the topology of $S^1 \times I$[1], which is Euclidean in nature. Therefore, when trying to apply such an invariant to a different geometric setting, say algebraic varieties, it does not play nicely with the very non-Euclidean Zariski topology.

Thus, we have to find a way to replicate the topological fundamental group in the setting of algebraic varieties. To compute the topological fundamental group one can use the theory of covering spaces. Which states that the topological fundamental group is isomorphic to the group of deck transformations of the universal cover. This can in fact be used as the definition of the topological fundamental group. This is a much more algebraic approach to defining the fundamental group. Thus, we can try to carry it over to algebraic geometry.

First, we will define the equivalent of covering maps, namely étale morphisms. We will find that we do not, in general, have a single universal cover. Instead, we have a system of covers which are universal in some sense. Taking the limit of the system of associated deck transformations yields us our étale fundamental group.

This process can be done in the setting of algebraic varieties, which allows us to compute the étale fundamental group of elliptic curves, and can also be done in the setting of schemes, which are a generalization of algebraic varieties and allow us to tackle a boarder family of geometric objects. We assume knowledge of both algebraic varieties and schemes, however a short introduction to schemes is given below. We also assume basic familiarity with commutative algebra. Any standard introductory text on commutative algebra covers more than enough, see for example [AM69].

Lastly, we will dive into étale cohomology. A full exposition of this theory is out of scope. We discuss a result relating the étale fundamental group and the first étale cohomology group.

[1]Where S^1 is the real unit circle and $I = [0, 1]$ the unit interval.

2 Preliminaries

2.1 Schemes, briefly

We recall the basics of scheme theory. In this document, any ring is commutative and unital unless otherwise specified, and homomorphisms preserve 1. If A is a ring, we write $\mathfrak{a} \trianglelefteq A$ to mean that $\mathfrak{a}$ is an ideal of A. The category of rings and ring homomorphisms is denoted **Ring**. If R is some ring, the category of R-algebras is denoted R-**Alg**.

A *ringed space* is a topological space X equipped with a sheaf of rings $\mathcal{O}_X$ on X. We denote the restriction homomorphisms of this sheaf by $\mathrm{res}_{U,V} \colon \mathcal{O}_X(U) \to \mathcal{O}_X(V)$ (where $V \subseteq U \subseteq X$ open) if it is necessary to name them. If $\varphi \colon X \to Y$ is a continuous map, the *pushforward* of $\mathcal{O}_X$ along φ is the sheaf $\varphi_*(\mathcal{O}_X)$ on Y given on open subsets $V \subseteq Y$ as $\varphi_*(\mathcal{O}_X)(V) = \mathcal{O}_X(\varphi^{-1}(V))$, and whose restriction homomorphisms are those of $\mathcal{O}_X$. A *morphism* between ringed spaces $X \to Y$ is a pair $(\varphi, \varphi^\sharp)$, where $\varphi \colon X \to Y$ is a continuous map and $\varphi^\sharp \colon \mathcal{O}_Y \to \varphi_*(\mathcal{O}_X)$ is a morphism of sheaves over Y.

Let X be a ringed space and let $p \in X$. The *stalk* of X at p is the ring

$$\mathcal{O}_{p,X} = \Big\{(U, f) : U \subseteq X \text{ open}, f \in \mathcal{O}_X(U)\Big\}\Big/\!\sim,$$

where $(U, f) \sim (V, g)$ whenever there exists an open neighborhood $W \subseteq U \cap V$ of p such that $\mathrm{res}_{U,W}(f) = \mathrm{res}_{V,W}(g)$. The class of (U, f) in $\mathcal{O}_{p,X}$ is denoted by $[U, f]$, or simply by f, where we think of f as having some 'flexible' domain of definition. Any morphism $(\varphi, \varphi^\sharp) \colon X \to Y$ of ringed spaces functorially induces homomorphisms between stalks $\varphi^\sharp_p \colon \mathcal{O}_{\varphi(p),Y} \to \mathcal{O}_{p,X}$.

A *locally ringed space* is a ringed space X such that at every point $p \in X$, the stalk $\mathcal{O}_{p,X}$ is a local ring. We write $\mathfrak{m}_{p,X}$ for the maximal ideal of $\mathcal{O}_{p,X}$. We will denote the stalks and maximal ideals by $\mathcal{O}_p$ and $\mathfrak{m}_p$ if the ambient space X is clear from context.

For local rings A and B with maximal ideals $\mathfrak{m}_A, \mathfrak{m}_B$, a *local homomorphism* from A to B is a ring homomorphism $\varphi \colon A \to B$ such that $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$. A *morphism of locally ringed spaces* is a morphism of ringed spaces $(\varphi, \varphi^\sharp) \colon X \to Y$ such that for all $p \in X$ we have that $\varphi^\sharp_p \colon \mathcal{O}_{\varphi(p)} \to \mathcal{O}_p$ is a local homomorphism. We will also call φ itself a morphism of locally ringed spaces, and unless otherwise specified we assume that its corresponding sheaf morphism is denoted $\varphi^\sharp$.

In the following, let A be any ring. We define a locally ringed space, called its spectrum, in several steps, following section II.2 in [Har77]. As a set, the *spectrum* of A is given by

$$\operatorname{Spec} A = \Big\{\mathfrak{p} \trianglelefteq A : \mathfrak{p} \text{ is prime}\Big\}.$$

For an ideal $\mathfrak{a} \trianglelefteq A$, define its corresponding *Zariski-open set* as

$$D_\mathfrak{a} = \Big\{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{a} \not\subseteq \mathfrak{p}\Big\}.$$

Subsets of $\operatorname{Spec} A$ of this form are closed under arbitrary unions and finite intersections, and they define the *Zariski topology* on $\operatorname{Spec} A$.

For $f \in A$, define its corresponding *distinguished open set* as

$$D_f = \Big\{\mathfrak{p} \in \operatorname{Spec} A : f \notin \mathfrak{p}\Big\}.$$

The distinguished open sets form a basis for the Zariski topology.

Let $S \subseteq A$ be a set of elements closed under multiplication, containing 1. The *localization* of A by S is the ring

$$S^{-1}A = \left\{ \frac{f}{g} : f \in A,\, g \in S \right\}.$$

Here fraction expressions f/g and f'/g' are identified if there exists $h \in S$ such that $h(fg' - f'g) = 0$. We consider two special cases. For a single element $f \in A$, the set $S = \{1, f, f^2, \dots\}$ is multiplicatively closed. The localization of A by S is denoted $A[f^{-1}]$. If $\mathfrak{p} \in \operatorname{Spec} A$ is a prime ideal, then $S = A \backslash \mathfrak{p}$ is multiplicatively closed, and the localization of A by S is denoted $A_\mathfrak{p}$, which we call the localization of A *at* $\mathfrak{p}$. Then $A_\mathfrak{p}$ is a local ring, with maximal ideal $\mathfrak{p}A_\mathfrak{p}$.

Let $U \subseteq \operatorname{Spec} A$ be open. A *regular function* on U is a function

$$f \colon U \to \bigsqcup_{\mathfrak{p} \in U} A_\mathfrak{p},$$

such that for all $\mathfrak{p} \in U$ we have that $f(\mathfrak{p}) \in A_\mathfrak{p}$, and f is locally a quotient of elements of A. More precisely, for every $\mathfrak{p} \in U$ there exists some open neighbourhood $V \subseteq U$ of $\mathfrak{p}$ and elements $g, h \in A$ such that for all $\mathfrak{q} \in V$ we have $h \notin \mathfrak{q}$ and $f(\mathfrak{q}) = g/h \in A_\mathfrak{q}$.

Let the *structure sheaf* $\mathcal{O}_{\operatorname{Spec} A}$ of $\operatorname{Spec} A$ be given as follows. Let $U \subseteq \operatorname{Spec} A$ be open. Define

$$\mathcal{O}_{\operatorname{Spec} A}(U) = \Big\{ f \colon U \to \bigsqcup_{\mathfrak{p} \in U} A_\mathfrak{p} : f \text{ is regular} \Big\}.$$

If $V \subseteq U$ are open subsets of $\operatorname{Spec} A$, define

$$\operatorname{res}_{U,V} \colon \mathcal{O}_{\operatorname{Spec} A}(U) \to \mathcal{O}_{\operatorname{Spec} A}(V)$$

by restriction. That is, for $\mathfrak{p} \in V$ and $f \in \mathcal{O}_{\operatorname{Spec} A}(U)$ we have $\operatorname{res}_{U,V}(f)(\mathfrak{p}) = f(\mathfrak{p})$. We write $f|_V = \operatorname{res}_{U,V}(f)$. This defines a sheaf of rings on the topological space $\operatorname{Spec} A$. Note that we have a well-defined evaluation homomorphism $\mathcal{O}_{\mathfrak{p}, \operatorname{Spec} A} \to A_\mathfrak{p}$ given by $[U, f] \mapsto f(\mathfrak{p})$.

We can alternatively represent the regular functions on distinguished opens and the stalks of regular functions of a spectrum as follows.

Proposition 2.1. *For $f \in A$ and $g \in A[f^{-1}]$, consider the regular function $c_g \colon D_f \to \bigsqcup_{\mathfrak{p} \in D_f} A_\mathfrak{p}$ that maps $\mathfrak{p} \in D_f$ to $g \in A_\mathfrak{p}$. Then the map $A[f^{-1}] \to \mathcal{O}_{\operatorname{Spec} A}(D_f)$ sending g to c_g is an isomorphism of rings.*

For $\mathfrak{p} \in \operatorname{Spec} A$, the evaluation map $\mathcal{O}_{\mathfrak{p}, \operatorname{Spec} A} \to A_\mathfrak{p}$ sending $[U, f]$ to $f(\mathfrak{p})$ is an isomorphism of rings.

Proof. See [Har77, proof of Proposition 2.2]. □

Definition 2.2. *An* affine scheme *is a locally ringed space $(X, \mathcal{O}_X)$ isomorphic to a spectrum $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A. A* scheme *is a locally ringed space $(X, \mathcal{O}_X)$ that is locally isomorphic to an affine scheme. That is, there exists some open cover (called an* affine cover*) $(U_i)_{i \in I}$ of X and rings $(A_i)_{i \in I}$ such that $(U_i, \mathcal{O}_{U_i}) \cong (\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$ for all $i \in I$.*

A morphism of schemes is a morphism of the underlying locally ringed spaces. The category of schemes is denoted **Sch**.

If $\alpha\colon A \to B$ is a ring homomorphism, we can define a corresponding scheme morphism $\varphi\colon \operatorname{Spec} B \to \operatorname{Spec} A$. As a continuous map, φ sends a prime ideal $\mathfrak{p} \trianglelefteq B$ to its preimage $\varphi(\mathfrak{p}) = \alpha^{-1}(\mathfrak{p})$, and the sheaf morphism $\varphi^\sharp$ sends a regular function $f\colon U \to \bigsqcup_{\mathfrak{p}\in U} A_\mathfrak{p}$ in $\mathcal{O}_{\operatorname{Spec} A}(U)$ to the function $\varphi^\sharp(f)\colon \varphi^{-1}(U) \to \bigsqcup_{\mathfrak{q}\in\varphi^{-1}(U)} B_\mathfrak{q}$, $\mathfrak{q} \mapsto \alpha_\mathfrak{q}(f(\varphi(\mathfrak{q})))$, where $\alpha_\mathfrak{q}\colon A_{\varphi(\mathfrak{q})} \to B_\mathfrak{q}$ is the local homomorphism obtained by localizing α at $\mathfrak{q}$.

Proposition 2.3. *The associations* $A \mapsto (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ *and* $\alpha \mapsto (\varphi, \varphi^\sharp)$ *define a fully faithful contravariant functor from* **Ring** *to* **Sch**.

Proof. See [Har77, Proposition 2.3]. □

In particular, **Ring** is antiequivalent to the full subcategory of **Sch** consisting of the affine schemes. It also follows that a morphism $\varphi\colon \operatorname{Spec} A \to \operatorname{Spec} B$ between affine schemes is fully determined by the component $\varphi^\sharp_{\operatorname{Spec} B}\colon \mathcal{O}_{\operatorname{Spec} B}(\operatorname{Spec} B) \to \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A)$, which by abuse of notation we also denote by $\varphi^\sharp$.

If X and Y are schemes, a *Y-valued point* (or a Y-point for short) of X is a morphism $Y \to X$. The set of Y-valued points of X is denoted $X(Y)$. A scheme morphism $\varphi\colon Y \to Z$ gives rise to a set function $X(Z) \to X(Y)$ functorially by precomposition. These associations together define the *functor of points* associated to X, and a functor naturally isomorphic to one of this form is called *representable*, with X its representing scheme. If A is some ring, an *A-valued point* of X is a $(\operatorname{Spec} A)$-valued point of X. We denote the A-valued points of X by $X(A)$. Composing the spectrum functor with the representable functor, we get that any ring homomorphism $\alpha\colon A \to B$ functorially induces a set function $X(A) \to X(B)$ (note that the composition of two contravariant functors is covariant). If it is necessary for clarity, we will refer to a point p of the topological space X as a *topological point* of the scheme X.

To round out the prerequisites, we cover relative schemes. Fix a scheme S. A scheme *over S* (or an *S-scheme*, for short) is a scheme X equipped with a morphism $\sigma\colon X \to S$. If R is a ring, a scheme over R is a scheme over $\operatorname{Spec} R$. Any scheme is a scheme over $\mathbb{Z}$ in a unique way. The structure morphism $\sigma\colon X \to \operatorname{Spec} R$ equips the ring of global regular functions $\mathcal{O}_X(X)$ with the structure of an R-algebra. If $R = k$ is a field, then the unique stalk of $\operatorname{Spec} k$ is canonically isomorphic to k by Proposition 2.1, so the stalk homomorphisms $\sigma^\sharp_p\colon k \to \mathcal{O}_{p,X}$ equip the stalks of X with the structure of a k-algebra.

A morphism of S-schemes from X to Y (also called an S-morphism) is a scheme morphism $\varphi\colon X \to Y$ such that the following diagram commutes,

$$\begin{array}{ccc} X & \xrightarrow{\ \varphi\ } & Y \\ & {\scriptstyle\sigma_X}\searrow \quad \swarrow{\scriptstyle\sigma_Y} & \\ & S & \end{array}$$

The category of S-schemes and S-morphisms is denoted S-**Sch**. If R is a ring and A is an R-algebra, the spectrum $\operatorname{Spec} A$ is an R-scheme in a canonical way, and the spectrum functor induces a fully faithful contravariant functor from R-**Alg** to R-**Sch**.

We also define a relative variant of the functor of points of a scheme. For S-schemes X, Y, define the Y-valued points of X relative to S as the set of S-scheme morphisms $Y \to X$, denoted $X_S(Y)$, or $X_A(Y)$ if $S = \operatorname{Spec} A$, or simply by $X(Y)$ if the context is clear. In the same way as in the non-relative case, any S-scheme X gives rise to a contravariant functor from S-**Sch** to **Set**. This finishes the prerequisites on scheme theory.

2.2 The topological fundamental group

There is an analogy between the topological fundamental group and the étale fundamental group. Thus, we first recall what the topological fundamental group is, and then move on and define the étale fundamental group.

Definition 2.4. *[Kun17, Definition 1.1] Let X be a real or complex manifold and $x_0 \in X$. The* topological fundamental group *is the set homotopy classes of maps from S^1 to X, with concatenation as its group operation. We denote it by*

$$\pi_1(X, x_0).$$

As stated in the introduction this invariant does not play well with algebraic varieties, since they come equipped with the Zariski topology. Consider the following example:

Example 2.5. Consider the ring $R = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$, the coordinate ring of the circle, $\operatorname{Spec} R$ will be the real unit circle. We expect to find $\pi_1(\operatorname{Spec} R, p) = \mathbb{Z}$, however this is not the case.

For any integral domain R and any $p \in \operatorname{Spec} R$, we have that $\pi_1(\operatorname{Spec} R, p) = 0$. Consider the loop $\gamma : I \to \operatorname{Spec} R$ which sends 0 and 1 to p and the rest to (0), it is straight forward to show that this is continuous and any other loop is homotopic to it. Giving us that the fundamental group is trivial.

Aside from the invariant not behaving the way we want it to, there is a moral argument to be made that this construction of invariant does not rely on objects/morphisms from the category of schemes or algebraic varieties. The morphisms this invariant is defined with exist in the category of topological spaces (up to homotopy), while the morphisms we consider carry information about the sheaves as well.

One approach to remedy this issue is by constructing a form of homotopy within algebraic geometry. This is what motivic homotopy theory concerns itself with, by replacing the the unit interval $[0, 1]$ with the affine line $\mathbb{A}^1$.

Another approach would be to see if we can define the fundamental group in such a way that we do not use homotopy but instead use techniques which can be more easily copied over into another category. Luckily, we can turn to covering theory to find the answer. This theory has its own 'niceness' conditions, so we will restrict ourselves to topological manifolds.

The main idea is as follows: to compute the fundamental group of a space X at a point x_0, we can look at larger spaces Y which 'cover' X. We can take a loop in X and try to lift it up into Y. However, its endpoints may end up in different places. It turns out, that this lifted path defines an automorphism on the fiber of x_0. Studying these transformations gives us information about the fundamental group of X.

We start by defining what a covering is.

Definition 2.6. *[Kun17, Definition 1.3] Let X and Y be manifolds and*

$$\psi : Y \to X$$

be a surjective map between them. Then the pair (Y, ψ) covers X *if given a point* $x \in X$*, and* $U \subset X$ *an open subset containing* x*, there exist open subsets* $U_\alpha \subset Y$ *such that*

$$\psi^{-1}(U) = \bigsqcup_\alpha U_\alpha$$

and moreover, $\psi|_{U_\alpha} : U_\alpha \to U$ *is a homeomorphism.*

Example 2.7. Consider $X = S^1$ as the unit circle of the complex plane. We have the covers $(S^1, z \mapsto z^n)$ and $(\mathbb{R}, t \mapsto e^{2\pi i t})$.

Definition 2.8. *For a space* X*,* $\mathrm{Cov}(X)$ *is the category of covers of* X*, whose morphisms are those morphisms between the covers which commute with projection.*

Proposition 2.9. *Let* X *be a manifold and* (Y, ψ) *a covering. Then for any loop* $\gamma : S^1 \to X$ *with* $\gamma(1) = x_0 \in X$ *and any* $y_0 \in \psi^{-1}(x_0)$*, there exists a unique path* $\tilde{\gamma} : I \to Y$ *with* $\tilde{\gamma}(0) = y_0$ *and* $\psi \circ \tilde{\gamma} = \gamma$*.*

Example 2.10. Let $\gamma : S^1 \to S^1$. Here we cover an example of a loop in S^1 getting lifted to a path in the helix, perhaps with an image?

As we have seen, this lift defines a function on the fiber of x_0

Proposition 2.11. *The function on the fiber induced by a loop* γ *depends only on the homotopy class of* γ*.*

This motivates us to look at the fundamental group of X, as this is the set of loops up to homotopy endowed with a group structure. It turns out that the induced functions respect this group structure. This is summarized in the following proposition.

Proposition 2.12. *The lifting functions define a group action where the fundamental group* $\pi_1(X, x_0)$ *acts on the fiber of* x_0*.*

Recall that a set together with a group action from the group G is called a G-set. We can summarize our findings in the following definition.

Definition 2.13. *The fiber functor* $\mathrm{Fib} : \mathrm{Cov}(X, x_0) \to \pi_1(X, x_0)-\mathbf{Set}$*, is defined to send a covering space* (Y, φ) *to the fiber* $\varphi^{-1}(x_0)$ *and a morphism of covers* α *to its restriction to the fibers.*

Now from the theory of covering spaces we know, given our 'niceness' conditions, of the existence of a universal cover.

Definition 2.14. *A universal cover* $(\widetilde{X}, \psi)$ *of* X *is a cover for which* $\widetilde{X}$ *is simply connected.*

This universal cover is unique up to isomorphism so we may sometimes refer to it as *the* universal cover. Aside from being unique, it has a very important property regarding its group of automorphisms $\mathrm{Aut}(\widetilde{X})$, where the automorphisms are of course taken to be in the category $\mathrm{Cov}(X)$.

Proposition 2.15. *[Hat03, Prop 1.39] For a universal cover* $(\widetilde{X}, \psi)$ *of* X*, we have that*

$$\mathrm{Aut}(\widetilde{X}) \cong \pi_1(X, x_0).$$

This definition of universal cover is in some sense still a very topological definition, therefore we would like to relate it to the more categorical machinery we have constructed, the fiber functor. To this end we introduce the following definition.

Definition 2.16. *A functor $F : \mathcal{C} \to \textbf{Set}$ is called* representable *if there exists an object $X \in \mathcal{C}$ such that for any object $Y \in \mathcal{C}$*

$$F(Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, Y)$$

in a natural way, i.e. the functors F and $\mathrm{Hom}_{\mathcal{C}}(X, \_)$ are naturally isomorphic. Here X is said to represent *F.*

Proposition 2.17. *[Mil13] Let $\widetilde{X}$ be a universal cover of X, and let $x_0 \in X$. Then* Fib *is represented by $\widetilde{X}$.*

With this we have managed to detach the topological parts from the definition of the fundamental group and rephrase it in a categorical fashion. This, in essence, is enough to define the fundamental group in a different setting, however, we will find that our fiber functor will not always be representable. Thus we will take it one step further and equate the topological fundamental group to the automorphisms of the functor itself.

Proposition 2.18. *[Mil13] Let $F : \mathcal{C} \to \textbf{Set}$ be a functor represented by $X \in \mathcal{C}$. Then*

$$\mathrm{Aut}_{\textbf{Set}^{\mathcal{C}}}(F) \cong \mathrm{Aut}_{\mathcal{C}}(X),$$

where F is taken in the category of functors from $\mathcal{C}$ to **Set**, *thus its automorphism are natural isomorphisms between itself.*

Which gives us our final result,

$$\mathrm{Aut}(\mathrm{Fib}) \cong \pi_1(X, x_0).$$

Which we shall take as our motivation to define the Étale fundamental group.

2.3 Étale morphisms

Having seen the construction of the topological fundamental group we will now try to mirror this definition using means in algebraic geometry. The first order of business is to define what will replace our coverings, for which we recall three definitions. First we discuss unramifiedness.

Definition 2.19. *[Mil13, page 20] Let A, B be local rings and $\psi : A \to B$ a local ring homomorphism. Let $\mathfrak{m}_A$ and $\mathfrak{m}_B$ be the maximal ideals of A and B respectively. We call ψ* unramified *if*

$$\psi(\mathfrak{m}_A)B = \mathfrak{m}_B$$

and $B/\mathfrak{m}_B$ is a finite separable field extension of $A/\mathfrak{m}_A$.

Definition 2.20. *[Mil13, page 20] A morphism of schemes $\psi : Y \to X$ is* unramified *if it is of finite type and the map*

$$\mathcal{O}_{\psi(y),X} \to \mathcal{O}_{y,Y}$$

is unramified for all $y \in Y$.

This is of course a very algebraic definition, however, it has a nice geometric interpretation.

Proposition 2.21. *Let $f : X \to Y$ be an unramified morphism of schemes and let $x \in X$ be any point. Then the induced map on tangent spaces $df_x : T_xX \to T_{f(x)}Y$ is injective.*

Next we discuss flatness.

Definition 2.22. *Let A be a ring. An A-module B is flat if the functor $(-) \otimes B$ is exact.*

Definition 2.23. *[Mil13, page 19] Let $\varphi : A \to B$ a ring homomorphism. We say φ is* flat *if B is a flat A-module. Here the A-module structure on B is induced by φ.*

Definition 2.24. *[Mil13, page 19] Let $\varphi : Y \to X$ be a morphism of schemes. We say φ is* flat *if*

$$\mathcal{O}_{\varphi(y),X} \to \mathcal{O}_{y,Y}$$

is flat for all $y \in Y$.

This is again a very algebraic definition, however geometrically, when varying over the base the fibers do not make sudden changes, i.e. for a flat morphism its fibers are all of the same dimension.

Combining these two definitions we get a morphism that induces not only an injection on tangent spaces, but in fact an isomorphism. Thus one can view the following definition as a 'local isomorphism' of sorts.

Definition 2.25. *[Mil13, page 20] A morphism of schemes is an* étale morphism *if it is a flat and unramified morphism. A morphism $\varphi^\sharp \colon A \to B$ of rings is étale if the associated map $\varphi \colon \operatorname{Spec} B \to \operatorname{Spec} A$ is étale.*

However we will restrict ourselves to finite étale morphisms.

Definition 2.26. *A morphism $f : X \to Y$ is called* finite *if there exists an open affine cover $\{U_i\}$ of Y such that each $V_i := f^{-1}(U_i)$ is affine and the induced map on rings*

$$\mathcal{O}_{U_i}(U_i) \to \mathcal{O}_{V_i}(V_i)$$

makes $\mathcal{O}_{V_i}(V_i)$ into a finitely generated module over $\mathcal{O}_{U_i}(U_i)$.

Which ensures that the fibers are finite sets, and thus that X 'covers' Y with a finite amount of sheets.

2.4 The Galois category and the étale fundamental group

Fix a field k, and a separable closure K of k with an embedding $k \hookrightarrow K$. Applying the spectrum functor to this embedding gets us a morphism $\sigma_K \colon \operatorname{Spec} K \to \operatorname{Spec} k$, giving $\operatorname{Spec} K$ the structure of a k-scheme. All schemes in this section are over k and all scheme morphisms are morphisms of k-schemes.

Definition 2.27. *Let S be a connected scheme. A* geometric point *of S is a K-point $\overline{s} \in S(K)$ (relative to k).*

Any morphism of schemes $\varphi \colon S \to T$ induces a function of sets $\varphi_* \colon S(K) \to T(K)$ given by $\varphi_*(\overline{s}) = \varphi \circ \overline{s}$. We call $\varphi_*(\overline{s})$ the *pushforward* of $\overline{s}$ along φ.

Fix a base connected scheme S and a geometric point $\overline{s} \in S(K)$. The pair $(S, \overline{s})$ is called a *pointed* scheme, analogous to a pointed topological space (X, x). To define the étale fundamental group of $(S, \overline{s})$, we will first construct its corresponding *Galois category*.

Definition 2.28. *A* finite étale cover *of S is a pair (X, e) consisting of a scheme X equipped with a finite étale morphism $e \colon X \to S$.*

We will often suppress the morphism e from the notation, so we refer to X as an étale cover of S. If X_α is some étale cover decorated with an index, then e_α will denote its defining étale morphism to S.

For an étale cover X, consider the set

$$F(X) = \left\{ \overline{x} \in X(K) : e_*(\overline{x}) = \overline{s} \right\}.$$

We call this the *fiber* of X over $\overline{s}$. If we want to specify the base point of S in the notation, we write $F_{\overline{s}}(X)$. A *pointed cover* of $(S, \overline{s})$ is a pair $(X, \overline{x})$ where X is an étale cover of S and $\overline{x} \in F(X)$.

For an étale cover X, let $\operatorname{Aut}(X)$ denote the group of S-automorphisms of (X, e). Then $\operatorname{Aut}(X)$ acts on $F(X)$ from the left, with the action given by

$$\psi \cdot \overline{x} = \psi_*(\overline{x}).$$

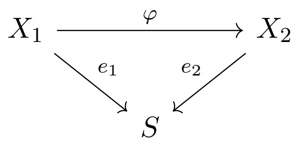
Proposition 2.29. *For any finite étale cover X of S, $F(X)$ is a finite set.*

Proof. This is generally true for fibers of finite morphisms. The general idea is that the points of $F(X)$ correspond to the points of the underlying topological space of the pullback scheme $X \times_S \operatorname{Spec} \kappa(\overline{s})$ (where $\kappa(\overline{s})$ is the residue field of S at the image of the geometric point $\overline{s}$), which is affine. Specifically, by finiteness it is the spectrum of a finite-dimensional algebra over k, which has finitely many prime ideals, so its spectrum finitely many points. □

Proposition 2.30. *If X is a connected finite étale cover of S, then $\operatorname{Aut}(X)$ acts freely on $F(X)$, meaning that if $\overline{x} \in F(X)$ and $\psi \in \operatorname{Aut}(X)$ are such that $\psi \cdot \overline{x} = \overline{x}$, then $\psi = \mathrm{id}_X$.*

Proof. [MA67, Lemma 4.4.1.6(iii)]. □

Definition 2.31. *Let the* Galois category $\mathbf{F\acute{E}t}/S$ *of S be the category whose objects are finite étale covers of S and whose morphisms from X_1 to X_2 are those scheme morphisms $\varphi\colon X_1 \to X_2$ such that*



commutes.

Let $\mathbf{F\acute{E}t}_*/S$ denote the category of pointed finite étale covers $(X, \overline{x})$ of S. The morphisms of this category are those morphisms of covers that preserve the base point. Note that we have an evident forgetful functor $\mathbf{F\acute{E}t}_*/S \to \mathbf{F\acute{E}t}/S$.

Let **FinSet** denote the category of finite sets.

Definition 2.32. *The* fiber functor *of $(S, \overline{s})$ is the functor $F\colon \mathbf{F\acute{E}t}/S \to \mathbf{FinSet}$ that maps a finite étale cover X onto the fiber $F(X)$, and a cover morphism $\varphi\colon X_1 \to X_2$ onto the induced function $\varphi_*\colon F(X_1) \to F(X_2)$.*

Definition 2.33. *The* étale fundamental group *of $(S, \overline{s})$, denoted $\pi_1(S, \overline{s})$, is defined as the automorphism group* $\operatorname{Aut}(F)$ *of the fiber functor F of $(S, \overline{s})$. That is, the set of natural isomorphisms from F to itself, with composition as group operation.*

Technically, as the Galois category $\mathbf{F\acute{E}t}/S$ is a large category, the collection of such natural isomorphisms is not a set. To avoid such set-theoretic size issues, we may assume that every scheme is contained in some set-theoretic universe U of 'small' sets, so that the above automorphism group is a 'large' set. We will not consider these issues in any more depth.

Explicitly, an element of $\pi_1(S, \bar{s})$ is a family of bijections $f_X \colon F(X) \to F(X)$ indexed by the finite étale covers $X \to S$, such that for any cover morphism $\varphi \colon X_1 \to X_2$ we have that

$$\begin{array}{ccc} F(X_1) & \xrightarrow{f_{X_1}} & F(X_1) \\ {\scriptstyle \varphi_*}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ F(X_2) & \xrightarrow{f_{X_2}} & F(X_2) \end{array}$$

commutes.

This description of $\pi_1(S, \bar{s})$ is not very practical, as a computation requires constructing natural bijections for every single possible finite étale cover. The rest of this section is devoted to a technique for calculating an explicit description of the fundamental group.

For calculation, let us first consider the case that the fiber functor F is a representable functor, as in the topological case. Let us call a finite étale cover $e \colon X \to S$ a *universal cover* if F is naturally isomorphic to $\mathrm{Hom}(X, -)$.

Proposition 2.34. *If F is represented by a finite étale cover $e \colon X \to S$, then*

$$\pi_1(S, \bar{s}) \cong \mathrm{Aut}(X).$$

Proof. By the Yoneda lemma, the natural transformations $\mathrm{Hom}(X, -) \Rightarrow \mathrm{Hom}(X, -)$ are in a natural bijection with the cover morphisms $X \to X$. It follows that the automorphism group of $\mathrm{Hom}(X, -)$ is isomorphic to that of X. Because F and $\mathrm{Hom}(X, -)$ are naturally isomorphic, their automorphism groups are isomorphic as well. $\square$

In general, such a representing object usually does not exist. We would still like to think of the fundamental group $\pi_1(S, \bar{s})$ as the automorphism group of some more general sort of 'universal cover'. Although this might not be a single finite étale cover, we can nonetheless construct a *system* of finite étale covers that will represent F.

Recall that we can view any preordered set $(\Lambda, \preceq)$ (i.e. a set equipped with a reflexive transitive relation) as a category, whose objects are the elements of Λ, and which has a unique arrow from i to j if and only if $i \preceq j$.

Definition 2.35. *A* directed set *is a preordered set Λ such that for all $i, j \in \Lambda$, there exists at least one $k \in \Lambda$ with $i \preceq k$ and $j \preceq k$.*

Definition 2.36. *Let $\mathbf{C}$ be a category, and let Λ be a directed set. An* inverse system *in $\mathbf{C}$ indexed by Λ is a functor $\mathcal{C} \colon \Lambda^{\mathrm{op}} \to \mathbf{C}$. Explicitly, a direct system associates to each $i \in \Lambda$ an object C_i in $\mathbf{C}$, and to each pair $i, j \in \Lambda$ with $i \preceq j$ a $\mathbf{C}$-morphism $\varphi_{ij} \colon C_j \to C_i$, satisfying $\varphi_{ii} = \mathrm{id}_{C_i}$ for all i, and if $i \preceq j \preceq k$, then the diagram*

$$\begin{array}{ccc} C_k & \xrightarrow{\varphi_{jk}} & C_j \\ & {\scriptstyle \varphi_{ik}}\searrow & \downarrow{\scriptstyle \varphi_{ij}} \\ & & C_i \end{array}$$

commutes. The morphisms φ_{ij} we call system morphisms*.*

We define inverse systems in order to talk about their limits. We will need the limits of inverse systems in the category of groups.

Definition 2.37. *Let $\mathcal{G} = (G_i)\colon \Lambda^{\mathrm{op}} \to \mathbf{Grp}$ be an inverse system of groups, with system homomorphisms denoted $h_{ij}\colon G_j \to G_i$. We define the* limit *of (G_i) as the group*

$$\varprojlim \mathcal{G} = \varprojlim G_i = \Big\{ (x_i) \in \prod_{i \in \Lambda} G_i : \text{if } i \preceq j, \text{ then } h_{ij}(x_j) = x_i \Big\}.$$

We equip $\varprojlim G_i$ with coordinatewise group operations.

Definition 2.38. *We say that an inverse system $(X_i, \overline{x}_i)\colon \Lambda^{\mathrm{op}} \to \mathbf{F\acute{E}t}_*/S$ is a* universal cover system *for $(S, \overline{s})$ if for every finite étale cover $e\colon X \to S$ there exists $i \in \Lambda$ and a cover morphism $\varphi_i\colon X_i \to X$.*

In order to calculate with universal cover systems, we need their covers to satisfy the following property.

Definition 2.39. *A* Galois cover *of S is a finite étale cover X such that X is connected and* $\mathrm{Aut}(X)$ *acts transitively on $F(X)$.*

Note that by Proposition 2.30, the action of $\mathrm{Aut}(X)$ on $F(X)$ is both free and transitive in the case that X is a Galois cover.

If $(X_1, \overline{x}_1), (X_2, \overline{x}_2)$ are pointed Galois covers of S, then a cover morphism $\varphi\colon X_1 \to X_2$ induces a group homomorphism $\widetilde{\varphi}\colon \mathrm{Aut}(X_1) \to \mathrm{Aut}(X_2)$ as follows. For $\psi \in \mathrm{Aut}(X_1)$, consider the point $\psi \cdot \overline{x}_1 \in F(X)$. By pushforward, φ sends this to $\varphi_*(\psi \cdot \overline{x}_1) \in F(Y)$. Because X_2 is a Galois cover, there exists a unique $\psi' \in \mathrm{Aut}(X_2)$ such that $\psi' \cdot \overline{x}_2 = \varphi_*(\psi \cdot \overline{x}_1)$. We define $\widetilde{\varphi}(\psi) = \psi'$. This does indeed provide a group homomorphism, though note that its definition is not independent of the choices of $\overline{x}_1, \overline{x}_2$.

This definition is functorial, so it gives a covariant functor Aut from the category of pointed Galois covers of S to the category of groups.

Theorem 2.40. *If $(X_i, \overline{x}_i)\colon \Lambda^{\mathrm{op}} \to \mathbf{F\acute{E}t}_*/S$ is a universal system of pointed Galois covers for $(S, \overline{s})$, then*

$$\pi_1(S, \overline{s}) \cong \varprojlim \mathrm{Aut}(X_i).$$

Proof. Consider an automorphism $\theta \in \mathrm{Aut}(F_{\overline{s}}) = \pi_1(S, \overline{s})$. For every finite étale cover X it has a component $\theta_X\colon F(X) \to F(X)$. For each $i \in \Lambda$, this provides a function $F(X_i) \to F(X_i)$. Using the fact that the X_i are Galois covers, let $\psi_i \in \mathrm{Aut}(X_i)$ be the unique automorphism such that $\psi_i \cdot \overline{x}_i = \theta_{X_i}(\overline{x}_i)$.

We claim that the indexed family $(\psi_i)_{i \in \Lambda}$ is an element of $\varprojlim \mathrm{Aut}(X_i)$. To see this, let $i \preceq j$, and consider the cover morphism $\varphi_{ij}\colon X_j \to X_i$. Because the covers are pointed and Galois, we have the induced group homomorphism $\widetilde{\varphi}_{ij}\colon \mathrm{Aut}(X_j) \to \mathrm{Aut}(X_i)$. We want to show that $\widetilde{\varphi}_{ij}(\psi_j) = \psi_i$. Consider where these automorphisms send $\overline{x}_i$. We have

$$\begin{aligned} \widetilde{\varphi}_{ij}(\psi_j) \cdot \overline{x}_i &= (\varphi_{ij} \circ \psi_j)_*(\overline{x}_j) = ((\varphi_{ij})_* \circ \theta_{X_j})(\overline{x}_j) \\ &= (\theta_{X_i} \circ (\varphi_{ij})_*)(\overline{x}_j) = \theta_{X_i}(\overline{x}_i) = \psi_i \cdot \overline{x}_i. \end{aligned}$$

Because X_i is connected, two automorphisms agreeing on a point implies that they are equal, so indeed $(\psi_i)_{i \in \Lambda} \in \varprojlim \mathrm{Aut}(X_i)$.

To see that this map $\pi_1(S,\overline{s}) \to \varprojlim \operatorname{Aut}(X_i)$ is a group homomorphism, simply note the following. If $\theta, \theta' \in \pi_1(S,\overline{s})$, and for $i \in \Lambda$ we have that $(\psi_i)_{i\in\Lambda}$ and $(\psi_i')_{i\in\Lambda}$ are the indexed families uniquely defined by $\psi_i \cdot \overline{x}_i = \theta_{X_i}(\overline{x}_i)$ and $\psi_i' \cdot \overline{x}_i = \theta'_{X_i}(\overline{x}_i)$ for all i, then

$$(\psi_i' \circ \psi_i) \cdot \overline{x}_i = (\theta' \circ \theta)_{X_i}(\overline{x}_i),$$

so our constructed map respects composition.

We need the fact that $(X_i, \overline{x}_i)$ is universal to show that this homomorphism is in fact an isomorphism. We will construct an inverse homomorphism. Let $(\psi_i)_{i\in\Lambda} \in \varprojlim \operatorname{Aut}(X_i)$ be some group element. To construct the natural automorphism $\theta \colon F \Rightarrow F$, let its components on the X_i be given by the action of the ψ_i, so $\theta_{X_i}(\overline{x}) = \psi_i \cdot \overline{x}$ for $\overline{x} \in F(X_i)$. To turn this into a natural automorphism of all of F, let $e \colon X \to S$ be an arbitrary connected finite étale cover. Applying universality, let $i \in \Lambda$ and $\varphi \colon X_i \to X$ be a cover morphism. By naturality, we have that

$$\begin{array}{ccc} F(X_i) & \xrightarrow{\theta_{X_i}} & F(X_i) \\ {\scriptstyle F(\varphi)}\downarrow & & \downarrow{\scriptstyle F(\varphi)} \\ F(X) & \xrightarrow{\theta_X} & F(X) \end{array}$$

commutes, so using the fact that X is connected we have that $\theta_X \colon F(X) \to F(X)$ is the unique bijection such that $\theta_X(F(\varphi)(\overline{x}_i)) = F(\varphi)(\psi_i \cdot \overline{x}_i)$. Now to extend θ to arbitrary (not necessarily connected) finite étale covers X, we use that fact that the connected components of X are themselves finite étale covers of S. The fact that $(\psi_i)_{i\in\Lambda}$ is a coherent sequence in the sense that $\widetilde{\varphi}_{ij}(\psi_j) = \psi_i$ for all system morphisms $\varphi_{ij} \colon X_j \to X_i$ implies that this is a well-defined natural transformation. We have constructed the required isomorphism. □

This theorem exhibits the fundamental group of $(S,\overline{s})$ as a profinite group, in the case that such a system $(X_i, \overline{x}_i)$ exists. As it turns out, we can always find such a system.

Proposition 2.41. *Let $(S,\overline{s})$ be a pointed scheme. Then there exists a universal system of pointed Galois covers for $(S,\overline{s})$.*

Proof. [MA67, Lemma 4.4.1.4]. □

Finally, it is worth mentioning that the fibers $F(X)$ for finite étale covers are not just mere sets; they come equipped with an obvious left group action by $\pi_1(S,\overline{s}$, and the induced functions between fibers are equivariant maps w.r.t. these actions. This leads us to state the following result. Let **DFPSet** denote the category of finite discrete $\pi_1(S,\overline{s})$-sets.

Theorem 2.42. *If $(S,\overline{s})$ is a connected pointed scheme, then the fiber functor*

$$F \colon \mathbf{F\acute{E}t}/S \to \mathbf{DFPSet}$$

is an equivalence of categories.

We will not prove this in its generality, but we prove it in the case of a point scheme $\operatorname{Spec} k$ at Theorem 3.30. The general case is treated in [Len08].

3 The example of a point

Let k be a field. In this section we investigate étale covers of $\operatorname{Spec} k$. We compute the étale fundamental group of $\operatorname{Spec} k$. Finally, we show that Theorem 2.42 holds in the case of $\operatorname{Spec} k$. Topologically, $\operatorname{Spec} k$ is a point. Thus, the topological fundamental group of $\operatorname{Spec} k$ is trivial. We show in Lemma 3.19 that the étale fundamental group of $\operatorname{Spec} k$ is $\operatorname{Aut}_k(k^{\mathrm{sep}})$, i.e. the absolute Galois group of k, but with addition given by $\psi_1 \cdot \psi_2 = \psi_2 \circ \psi_1$ for all $\psi_1, \psi_2 \in \operatorname{Aut}_k(k^{\mathrm{sep}})$. The group $\operatorname{Aut}_k(k^{\mathrm{sep}})$ has a rich structure. This further illustrates why we consider the étale fundamental group in addition to the topological fundamental group when investigating schemes. This section is based on the examples on page 22 and the examples on page 28 and 29 of [Mil13]. We first discuss some commutative algebra and introduce notation.

3.1 Étale covers of a point

Lemma 3.1. *Let M be a k-module. Then M is flat.*

Proof. Note that k, 0 are flat k-modules. By [AM69, Exercise 2.24], $\operatorname{Tor}_1^k(k, M) = \operatorname{Tor}_1^k(0, M) = 0$. Since $(1), (0)$ are the only ideals of k, [AM69, Exercise 2.26] gives that M is flat. □

Remark 3.2. *Let R be a local ring and $\varphi : k \to R$ a morphism of rings. Then φ is a local morphism of rings. From Definition 2.19, it follows that φ is unramified if and only if R is a finite separable field extension of k.*

Notation 3.3. Let $A_1, \dots, A_n$ be local k-algebras for some $n \in \mathbb{N}_{>0}$. For all $1 \leq j \leq n$, we denote by $\mathfrak{m}_j$ the maximal ideal of A_j. By $\mathfrak{m}'_j$ we denote the maximal ideal $A_1 \times \dots \times A_{j-1} \times \mathfrak{m}_j \times A_{j+1} \times \dots \times A_n$ of $A_1 \times \dots \times A_n$. When we talk about a product of k-algebras, we always mean a product in the category of k-algebras, so the product carries a k-algebra structure. If it is clear from context what the indices of a tuple in $A_1 \times \dots \times A_n$ are, we do not state this explicitly. For example, if $a \in A_j$, then $(0 \dots, 0, a, 0, \dots, 0)$ denotes the tuple where the j-th component is a.

Remark 3.4. *Let $R_1, \dots, R_n$ be rings for some $n \in \mathbb{N}_{>0}$. The prime ideals of $R_1 \times \dots \times R_n$ are exactly the ideals $R_1 \times \dots \times R_{j-1} \times \mathfrak{p} \times R_j \times \dots \times R_n$ for all $1 \leq j \leq n$ and all prime ideals $\mathfrak{p} \subseteq R_j$. The maximal ideals are exactly the ideals $R_1 \times \dots \times R_{j-1} \times \mathfrak{m} \times R_j \times \dots \times R_n$ for all $1 \leq j \leq n$ and all maximal ideals $\mathfrak{m} \subseteq R_j$.*

Below, we characterize all étale covers of $\operatorname{Spec} k$.

Lemma 3.5. *Let $\varphi : k \to R$ be a ring morphism. Then φ is étale if and only if R is $\{0\}$ or R is a finite product of finite separable field extensions of k.*

Proof. Note that φ is étale if $R = \{0\}$. Assume $R \neq \{0\}$. Assume that φ is étale. Let $\mathfrak{m} \subseteq R$ be a maximal ideal. Then $R_\mathfrak{m}$ is a finite separable field extension of k. This implies that $R_\mathfrak{m}$ has Krull dimension zero. Assume there exists a chain of prime ideals $\mathfrak{p} \subsetneq \mathfrak{m} \subsetneq R$. Then $\dim R_\mathfrak{m} > 0$. This gives a contradiction. We conclude that $\dim R = 0$. Since R is a finite type k-algebra, it is Noetherian. We conclude that R is Artin, since it is Noetherian of dimension 0. Thus $R = A_1 \times \dots \times A_n$ is a finite product of local Artin rings. Consider the ring morphism $\psi : R \to A_j \to (A_j)_{\mathfrak{m}_j}$, where $R \to A_j$ is the j-th projection map, and $A_j \to (A_j)_{\mathfrak{m}_j}$ is the map from the construction of the localization. Consider A_j and $(A_j)_{\mathfrak{m}_j}$ as k-algebras, where the k-algebra structure is given by the composition of these maps and

φ. Let $(a_1, \dots, a_n) \in R \backslash \mathfrak{m}'_j$. Then $a_j \notin \mathfrak{m}_j$, so $\psi((a_1, \dots, a_n)) \in (A_j)^{\times}_{\mathfrak{m}_j}$. The universal property of the localization gives a k-algebra morphism $\tilde{\psi} : R_{\mathfrak{m}'_j} \to (A_j)_{\mathfrak{m}_j}$. Let $\frac{a}{b} \in (A_j)_{\mathfrak{m}_j}$. Then $\tilde{\psi}(\frac{(1,\dots,1,a,1,\dots,1)}{(1,\dots,1,b,1,\dots,1)}) = \frac{a}{b}$. We conclude that $\tilde{\psi}$ is surjective. Let $\frac{(a_1,\dots,a_n)}{(b_1,\dots,b_n)} \in R_{\mathfrak{m}'_j}$ such that $\tilde{\psi}(\frac{(a_1,\dots,a_n)}{(b_1,\dots,b_n)}) = 0$. Then there exists $b'_j \in A_j \backslash \mathfrak{m}_j$ such that $a_j b'_j = 0 \in A_j$. Then $(0, \dots, 0, b'_j, 0, \dots, 0) \in R \backslash \mathfrak{m}'_j$ and $(a_1, \dots, a_n)(0, \dots, 0, b'_j, 0, \dots, 0) = 0 \in R$. This implies that $\tilde{\psi}$ is injective. Since $\tilde{\psi}$ is an isomorphism and $R_{\mathfrak{m}'_j}$ is a finite separable field extension of k, $(A_j)_{\mathfrak{m}_j}$ is a finite separable field extension of k. Let $a \in \mathfrak{m}_j$. Then $\frac{a}{1}$ is an element of the maximal ideal of $(A_j)_{\mathfrak{m}_j}$, so $\frac{a}{1} = 0 \in (A_j)_{m_j}$. Then there exists $b \in A_j \backslash \mathfrak{m}_j$ such that $ab = 0$. This gives $a = 0 \cdot b^{-1} = 0$. We conclude that $\mathfrak{m}_j = (0)$, so A_j is a field. In particular, $A_j = (A_j)_{\mathfrak{m}_j}$ is a finite separable field extension of k. We have shown that R is a finite product of finite separable field extensions of k.
Conversely, assume that $R = k_1 \times \ldots \times k_n$ for $k_1, \dots, k_n$ finite separable field extensions of k and $n \in \mathbb{N}_{>0}$. By Lemma 3.1, φ is flat. Since $k_1, \dots, k_n$ are finite k-modules, φ is of finite type. Consider the j-th projection map $\pi_j : R \to k_j$. Let $(a_1, \dots, a_n) \in R \backslash \mathfrak{m}'_j$. Then $\pi_j((a_1, \dots, a_n)) = a_j \in k_j^{\times}$. The universal property of the localisation gives a k-algebra morphism $\tilde{\pi_j} : R_{\mathfrak{m}'_j} \to k_j$. Observe that $\tilde{\pi_j}$ is surjective. Let $\frac{(a_1,\dots,a_n)}{(b_1,\dots,b_n)} \in R_{\mathfrak{m}'_j}$ such that $\tilde{\pi_j}(\frac{(a_1,\dots,a_n)}{(b_1,\dots,b_n)}) = a_j b_j^{-1} = 0$. Thus $a_j = 0$. Then

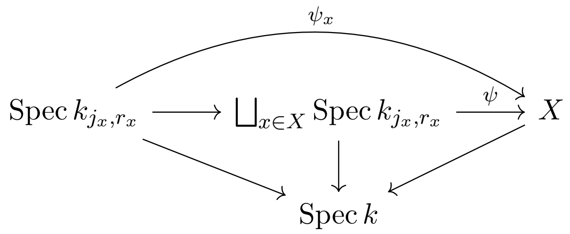
$$\frac{(a_1, \dots, a_n)}{(b_1, \dots, b_n)} = \frac{(a_1, \dots, a_{j-1}, 0, a_{j+1}, \dots, a_n)}{(b_1, \dots, b_n)} = 0.$$

Here we use $(a_1, \dots, a_{j-1}, 0, a_{j+1}, \dots, a_n)(0, \dots, 0, 1, 0, \dots, 0) = 0 \in R$ and $(0, \dots, 0, 1, 0, \dots, 0) \in R \backslash \mathfrak{m}'_j$. We conclude that $\tilde{\pi_j}$ is an isomorphism. Since k_j is a finite separable field extension of k, $R_{\mathfrak{m}'_j}$ is a finite separable field extension of k. This gives that R is unramified over k. We have shown that R is étale over k. □

Lemma 3.6. *Let X be a scheme and $\varphi : X \to \operatorname{Spec} k$ be an étale morphism. Then as a k-scheme, $X = \operatorname{Spec}\{0\}$ or $X = \operatorname{Spec} R$ where R is a finite product of finite separable field extensions of k.*

Proof. Note that φ is étale if X is empty. Assume that $X \neq \emptyset$. Let $U \subseteq X$ be a nonempty affine open and denote $U = \operatorname{Spec} R$. Note that $\varphi|_U : \operatorname{Spec} R \to \operatorname{Spec} k$ is of finite type. The other requirement for a scheme morphism to be unramified and flatness are local properties, so $\varphi|_U$ is étale. It follows that the associated ring morphism $\varphi^\sharp : k \to R$ is étale. Lemma 3.5 implies that $R = k_1 \times \ldots \times k_n$ for $k_1, \dots, k_n$ finite separable field extensions of k and some $n \in \mathbb{N}_{>0}$. Since φ is étale, it is of finite type, so X is quasi-compact. Because of this, we can cover X by a finite number of nonempty affine opens $U_1, \dots, U_r$ such that $U_j = \operatorname{Spec} R_j$ and R_j is a finite product of finite separable field extensions of k for all $1 \leq j \leq r$. Let $x \in X$, then choose some U_{j_x} such that $x \in U_{j_x}$. Note that $U_{j_x} = \operatorname{Spec} R_{j_x} = \operatorname{Spec}(k_{j_x,1} \times \ldots \times k_{j_x,n_{j_x}}) = \bigsqcup_{1 \leq s \leq n_{j_x}} \operatorname{Spec} k_{j_x,s}$ where $k_{j_x,1}, \dots, k_{j_x,n_{j_x}}$ are finite separable field extensions of k and $n_{j_x} \in \mathbb{N}_{>0}$ some positive integer. Consider the morphism of k-schemes $\psi_x : \{x\} = \operatorname{Spec} k_{j_x,r_x} \to U_{j_x} \to X$. Here $\operatorname{Spec} k_{j_x,r_x} \to U_{j_x}, U_{j_x} \to X$ denote the inclusion maps. By the universal property of the co-product of schemes (see [FdJ23, Exercise 3.4.iii]), there exists a morphism of schemes $\psi : \bigsqcup_{x \in X} \operatorname{Spec} k_{j_x,r_x} \to X$ such that $\psi|_{\operatorname{Spec} k_{j_x,r_x}} = \psi_x$ for all $x \in X$. Note that ψ is a bijection by construction. Let $x \in X$, then $x \cap U_j$ is closed for all $1 \leq j \leq n$. This follows since every U_j is a finite union of open and closed points. Thus X has the discrete topology. It follows that ψ is a homeomorphism. Cover X by opens $\{x\} \subseteq X$ for all $x \in X$.

Then $\psi^\sharp(\{x\}) : \mathcal{O}_X(\{x\}) \to \mathcal{O}_{\bigsqcup_{x\in X} \operatorname{Spec} k_{j_x,r_x}}(\operatorname{Spec} k_{j_x,r_x})$ is an isomorphism. This follows since $\mathcal{O}_X(\{x\}) = \mathcal{O}_{U_{j_x}}(\{x\}) = k_{j_x,r_x}$ and $\mathcal{O}_{\bigsqcup_{x\in X} \operatorname{Spec} k_{j_x,r_x}}(\operatorname{Spec} k_{j_x,r_x}) = k_{j_x,r_x}$. Thus ψ is an isomorphism of schemes. For all $x \in X$, consider the following diagram.



The universal property of the co-product gives that the bottom left triangle and the top triangle commute. Since ψ_x is a morphism of k-schemes, the outer triangle commutes. Since the $\operatorname{Spec} k_{j_x,r_x}$ form an open cover of $\bigsqcup_{x\in X} \operatorname{Spec} k_{j_x,r_x}$, it follows that the bottom right triangle commutes. We conclude that $X = \bigsqcup_{x\in X} \operatorname{Spec} k_{j_x,r_x} = \operatorname{Spec}(\prod_{x\in X} k_{j_x,r_x})$ as a k-scheme. $\square$

Remark 3.7. *Lemma* 3.6 *implies that all étale covers of* $\operatorname{Spec} k$ *are finite.*

3.2 Étale fundamental group of a point

Notation 3.8. In this section, Y, Y_1 and Y_2 denote étale covers of $\operatorname{Spec} k$ unless specified otherwise. The corresponding rings are denoted by R, R_1, R_2 respectively. If Y, Y_1, Y_2 are nonempty we use the notation $R = k_1 \times \ldots \times k_n, R_1 = k_1 \times \ldots \times k_{n_1}, R_2 = k'_1 \times \ldots \times k'_{n_2}$ where $k_1, \ldots, k_n, k_{n_1}, k'_1, \ldots, k'_{n_2}$ are finite separable field extensions of k and $n, n_1, n_2 \in \mathbb{N}_{>0}$. As in Notation 3.3, the maximal ideals of R, R_1 are denoted by $\mathfrak{m}'_j$ and the maximal ideals of R_2 are denoted by $\mathfrak{n}'_j$.

Let Ω be a separable closure of k and denote by $\overline{s}\colon \operatorname{Spec}\Omega \to \operatorname{Spec} k$ the geometric point associated to $k \subseteq \Omega$. We will now work toward the construction of $\pi_1(\operatorname{Spec} k, \overline{s})$. First, we cover some technical results.

Lemma 3.9. *Let $k \subseteq k_1, \ldots, k_n \subseteq \Omega$ be finite separable field extensions of k for some $n \in \mathbb{N}_{>0}$. Then there exists a finite Galois extension $k \subseteq L \subseteq \Omega$ such that $k_1, \ldots, k_n \subseteq L$.*

Proof. Since $k_1, \ldots, k_n$ are finite separable field extensions of k, there exist $\alpha_1, \ldots, \alpha_n \in \Omega$ such that $k_1 = k(\alpha_1), \ldots, k_n = k(\alpha_n)$. The field extension $k(\alpha_1, \ldots, \alpha_n)$ is finite and separable over k. Thus there exists $\gamma \in \Omega$ such that $k(\alpha_1, \ldots, \alpha_n) = k(\gamma)$. It follows that the splitting field over k of the minimal polynomial of γ over k is a finite Galois extension of k that contains $k_1, \ldots, k_n$. $\square$

Remark 3.10. *Let $k_1, \ldots, k_n$ be as in Lemma* 3.9*. The finite Galois extension constructed in the proof of Lemma* 3.9 *is the smallest Galois extension of k containing $k_1, \ldots, k_n$.*

Notation 3.11. We denote the image of a morphism by Im.

Lemma 3.12. *Let $\psi\colon k_1 \times \ldots \times k_n \to \Omega$ be a k-algebra morphism. Then ψ factors through $k_1 \times \ldots \times k_n/\mathfrak{m}'_j = k_j$ for some $1 \le j \le n$.*

Proof. Since Ω is a field, $\operatorname{Im}(\psi)$ is an integral domain. It follows that $\ker(\psi)$ is a prime ideal. This implies that $\ker(\psi) = \mathfrak{m}'_j$ for some $1 \le j \le n$. Since $k_1 \times \ldots \times k_n/\mathfrak{m}'_j = k_j$ the result follows. $\square$

Corollary 3.12.1. *Let $k_1, \ldots, k_n$ be finite separable field extensions of k for some $n \in \mathbb{N}_{>0}$. Let $\psi\colon k_1 \times \ldots \times k_n \to \Omega$ be a k-algebra morphism. Then $\operatorname{Im}(\psi)$ is contained in a finite Galois extension of k.*

Proof. From Lemma 3.12, it follows that $\operatorname{Im}(\psi)$ is a finite separable extension of k. Lemma 3.9 implies that $\operatorname{Im}(\psi)$ is contained in a finite Galois extension of k. $\square$

Consider the set

$$I := \{L \subseteq \Omega \mid k \subseteq L \text{ is a finite Galois extension}\}.$$

We say that $L_1 \leq L_2$ for $L_1, L_2 \in I$ if $L_1 \subseteq L_2$. Lemma 3.9 implies that there exists some $L \in I$ such that $L_1, L_2 \leq L$. We conclude that $(I, \leq)$ is a directed set. Let $L_1, L_2 \in I$ such that $L_1 \leq L_2$. The inclusion $\varphi^\sharp_{L_1,L_2} : L_1 \hookrightarrow L_2$ gives a morphism of schemes over $\operatorname{Spec} k$, denoted $\varphi_{L_1,L_2} : \operatorname{Spec} L_2 \to \operatorname{Spec} L_1$. Let Y be some scheme over $\operatorname{Spec} k$. We obtain a morphism of sets $\varphi^*_{L_1,L_2} : \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L_1, Y) \to \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L_2, Y)$ sending g to $g \circ \varphi_{L_1,L_2}$. Observe that the sets $\operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y)$ and morphisms $\varphi^*_{L_1,L_2}$ form a direct system over I.

Notation 3.13. Let $L' \in I$, Y a scheme over $\operatorname{Spec} k$ and $\psi \in \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L', Y)$. We denote the image of ψ under the canonical map $\operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L', Y) \to \varinjlim_{L \in I} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y)$ by $[\psi]$. For such $[\psi] \in \varinjlim_{L \in I} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y)$, when it is clear from context, we don't specify to what $\operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L', Y)$ the ψ initially belonged.

Notation 3.14. Let $\xi\colon Y_1 \to Y_2$ be morphism of schemes over $\operatorname{Spec} k$. We denote the induced map $\varinjlim_{L \in I} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y_1) \to \varinjlim_{L \in I} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y_2)$ by $\xi \circ -$.

Proposition 3.15. *The functor $F\colon \mathbf{F\acute{E}t}/\operatorname{Spec} k \to \mathbf{Set}$ is prorepresented by the direct system over I constructed above, where the sets are given by $\operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y)$ and the morphisms are given by $\varphi^*_{L_1,L_2}$ for all $L \in I$ and all $L_1, L_2 \in I$ such that $L_1 \leq L_2$.*

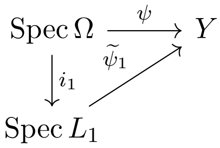
Proof. If $Y = \emptyset$, then $F(Y) = \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, Y) = \emptyset$. On the other hand

$$\varinjlim_{L \in I} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y) = \varinjlim_{L \in I} \emptyset = \emptyset.$$

In this case, we denote by $\gamma(Y)\colon F(Y) \to \varinjlim_{L \in I} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y)$ the unique bijection. Let $Y \to \operatorname{Spec} k$ be a nonempty étale cover of $\operatorname{Spec} k$. Then we have bijections $F(Y) = \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, Y) = \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(\mathcal{O}_Y(Y), \Omega)$. Let $\psi\colon \operatorname{Spec} \Omega \to Y$ be a morphism over $\operatorname{Spec} k$. By Lemma 3.6, $\mathcal{O}_Y(Y)$ is isomorphic to a finite product of finite separable field extensions of k. By Corollary 3.12.1, there exists a finite Galois extension of k, $k \subseteq L_1 \subseteq \Omega$ such that $\operatorname{Im}(\psi^\sharp) \subseteq L_1$. We obtain a commutative triangle where $i_1^\sharp\colon L_1 \to \Omega$ denotes the inclusion map.

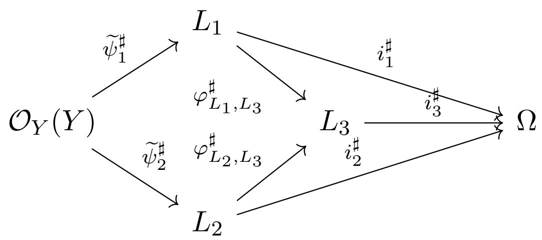
$$\begin{array}{ccc} \mathcal{O}_Y(Y) & \xrightarrow{\psi^\sharp} & \Omega \\ \big\downarrow \widetilde{\psi}_1^\sharp & \nearrow_{i_1^\sharp} & \\ L_1 & & \end{array}$$

From this, we get the following commutative triangle.



(1)

Note that $\widetilde{\psi}_1$ is a morphism over $\operatorname{Spec} k$. We define a map $\gamma(Y)\colon \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec}\Omega, Y) \to \varinjlim_{L\in I} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y)$ by sending ψ to the equivalence class $[\widetilde{\psi}_1]$. We show that $\gamma(Y)$ is a well-defined bijection. Let $k \subseteq L_2 \subseteq \Omega$ be another finite Galois extension of k such that $\operatorname{Im}(\psi^\sharp) \subseteq L_2$. By Lemma 3.9, there exists a finite Galois extension of k, $L_3 \subseteq \Omega$ such that $L_1, L_2 \subseteq L_3$. Using similar notation as before, we obtain the following diagram.

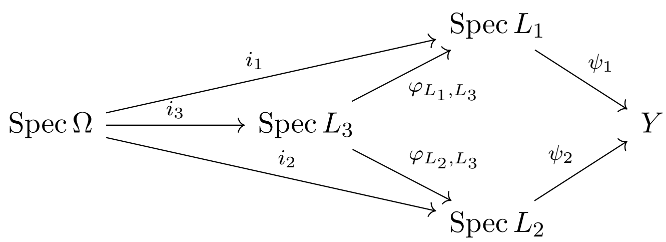


It follows that $\varphi^\sharp_{L_2,L_3} \circ \widetilde{\psi}^\sharp_2 = \varphi^\sharp_{L_1,L_3} \circ \widetilde{\psi}^\sharp_1$. This gives

$$\varphi^*_{L_2,L_3}(\widetilde{\psi}_2) = \widetilde{\psi}_2 \circ \varphi_{L_2,L_3} = \widetilde{\psi}_1 \circ \varphi_{L_1,L_3} = \varphi^*_{L_1,L_3}(\widetilde{\psi}_1).$$

Thus $[\widetilde{\psi}_1] = [\widetilde{\psi}_2] \in \varinjlim_{L\in I} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y)$. We conclude that $\gamma(Y)$ is well-defined.

To show that $\gamma(Y)$ is bijective, we construct an inverse. Let $L_1 \in I$, using similar notation as before, $i_1^*\colon \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L_1, Y) \to \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec}\Omega, Y)$ sending ψ to $\psi \circ i_1$ is a well-defined map of sets. Consider the map $\zeta\colon \varinjlim_{L\in I} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y) \to \operatorname{Hom}_{\operatorname{Spec} k}(\Omega, Y)$ given by sending $[\psi_1]$ to $\psi_1 \circ i_1$. Here $[\psi_1]$ denotes the equivalence class in $\varinjlim_{L\in I} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y)$ of an element $\psi_1 \in \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L_1, Y)$. We show that ζ is well-defined. Let $\psi_1 \in \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L_1, Y), \psi_2 \in \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L_2, Y)$ such that there exists $L_3 \in I$, where $L_1, L_2 \subseteq L_3$ and $\varphi^*_{L_1,L_3}(\psi_1) = \varphi^*_{L_2,L_3}(\psi_2)$. Consider the following diagram.



The triangles on the left commute and the small square on the right commutes. It follows that the outer diamond commutes. Because of this, ζ is well-defined. We now show that it is an inverse of $\gamma(Y)$. Let $L_1 \in I$ and $\psi_1 \in \operatorname{Hom}_{\operatorname{Spec} k}(L_1, Y)$. Then $\zeta([\psi_1]) = \psi_1 \circ i_1$. From the construction of $\gamma(Y)$, it follows that $\gamma(Y)(\psi_1 \circ i_1) = [\psi_1]$. Let $\psi \in \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec}\Omega, Y)$. From the construction of $\gamma(Y)$, we obtain a commutative diagram as in (1). Then $\zeta \circ \gamma(Y)(\psi) = \zeta([\widetilde{\psi}_1]) = \widetilde{\psi}_1 \circ i_1 = \psi$. We conclude that $\gamma(Y)$ is a bijection. Let $\xi : Y_1 \to Y_2$ be a morphism of étale covers of $\operatorname{Spec} k$ and assume that $Y_1 \neq \emptyset$. Let $[\psi_3] \in \varinjlim_{L\in I} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y_1)$. Then $\xi \circ \gamma(Y_1)^{-1}([\psi_3]) = \xi \circ \psi_3 \circ i_3 = \gamma(Y_2)^{-1}(\xi \circ \psi_3)$. We conclude that the following diagram commutes.

$$\begin{array}{ccc} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, Y_1) & \xrightarrow{\gamma(Y_1)} & \varinjlim_{L\in I} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y_1) \\ \downarrow{\scriptstyle \xi\circ -} & & \downarrow{\scriptstyle \xi\circ -} \\ \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, Y_2) & \xrightarrow{\gamma(Y_2)} & \varinjlim_{L\in I} \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} L, Y_2) \end{array} \tag{2}$$

Observe that if $Y_1 = \emptyset$, then a diagram similar to the one in (2) also commutes. This finishes the proof. □

From now on, the multiplication on $\operatorname{Aut}_k(L)$ is given by $\psi_1^\sharp \cdot \psi_2^\sharp = \psi_2^\sharp \circ \psi_1^\sharp$ for all $L \in I$. Observe that this is indeed a group. Then $\operatorname{Aut}_k(L) \times \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(L, \Omega) \to \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(L, \Omega)$ given by sending $(\psi^\sharp, g^\sharp)$ to $g^\sharp \circ \psi^\sharp$ defines a left-action on $\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(L, \Omega)$.

Lemma 3.16. *For all $L \in I$, the left $\operatorname{Aut}_k(L)$-action on $\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(L, \Omega)$ defined above is free and transitive.*

Proof. Let $L \in I$. Since L is a finite Galois extension of k, $\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(L, \Omega)$ can be identified with $\operatorname{Aut}_k(L)$. Let $g_1^\sharp, g_2^\sharp \in \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(L, \Omega)$. Then $g_1^\sharp \circ (g_1^\sharp)^{-1} \circ g_2^\sharp = g_2^\sharp$. We conclude that the action is transitive. Let $g^\sharp \in \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(L, \Omega), \psi^\sharp \in \operatorname{Aut}_k(L)$ such that $g^\sharp \circ \psi^\sharp = g^\sharp$. Then $\psi^\sharp = (g^\sharp)^{-1} \circ g^\sharp = \operatorname{Id}_L$. It follows that the action is free. □

Lemma 3.17. *For all $L \in I$, denote by $\operatorname{Spec} L \to \operatorname{Spec} k$ the map associated to the inclusion $k \subseteq L \subseteq \Omega$. Then $\operatorname{Spec} L \to \operatorname{Spec} k$ is a Galois cover of $\operatorname{Spec} k$.*

Proof. From Lemma 3.5, it follows that $\operatorname{Spec} L \to \operatorname{Spec} k$ is an étale cover of $\operatorname{Spec} k$. Consider the following diagram.

$$\begin{array}{ccc} \operatorname{Aut}_{\operatorname{Spec} k}(\operatorname{Spec} L) \times \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, \operatorname{Spec} L) & \longrightarrow & \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, \operatorname{Spec} L) \\ \downarrow & & \downarrow \\ \operatorname{Aut}_k(L) \times \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(L, \Omega) & \longrightarrow & \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(L, \Omega) \end{array}$$

The map $\operatorname{Aut}_{\operatorname{Spec} k}(\operatorname{Spec} L) \times \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, \operatorname{Spec} L) \to \operatorname{Aut}_k(L) \times \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(L, \Omega)$ is given by sending (ψ, g) to $(\psi^\sharp, g^\sharp)$. The horizontal maps are the relevant group actions. The map $\operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, \operatorname{Spec} L) \to \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(L, \Omega)$ sends g to $g^\sharp$. Note that the map $\operatorname{Aut}_{\operatorname{Spec} k}(\operatorname{Spec} L) \to \operatorname{Aut}_k(L)$ given by sending ψ to $\psi^\sharp$ is an isomorphism of groups because of the unusual group structure on $\operatorname{Aut}_k(L)$. Also note that the vertical maps are bijections and that the diagram commutes. From Lemma 3.16, it follows that the $\operatorname{Aut}_{\operatorname{Spec} k}(\operatorname{Spec} L)$-action on $\operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, \operatorname{Spec} L)$ is free and transitive. □

During the construction of I, we chose embeddings $k \subseteq L \subseteq \Omega$ for every $L \in I$. These give base points $\operatorname{Spec} \Omega \to \operatorname{Spec} L$ for all $L \in I$. Proposition 3.15 and Lemma 3.17 imply that we have constructed a based universal cover of $\operatorname{Spec} k$. Because of this and Theorem 2.40, we obtain an étale fundamental group of $\operatorname{Spec} k$. We will now give a characterisation of this étale fundamental group as $\mathrm{Aut}_k(\Omega)$. In the remainder of this section, we assume that the reader is familiar with Galois theory of infinite extensions. See for example [Mil22, *Chap.*7].

Consider $k \subseteq L_1 \subseteq L_2 \subseteq \Omega$ for $L_1, L_2 \in I$. Denote the inclusions $\varphi^\sharp_{L_1,L_2} \colon L_1 \to L_2, i^\sharp_1 \colon L_1 \to \Omega, i^\sharp_2 \colon L_2 \to \Omega$. This gives rise to the following diagram.

$$\begin{array}{ccccc} \mathrm{Aut}_k(L_2) & \longrightarrow & \mathrm{Aut}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L_2) & \xrightarrow{-\circ i_2} & \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\,\Omega, \mathrm{Spec}\, L_2) \\ \Big\downarrow |_{L_1} & & & & \Big\downarrow \varphi_{L_1,L_2}\circ - \\ \mathrm{Aut}_k(L_1) & \longrightarrow & \mathrm{Aut}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L_1) & \xrightarrow{-\circ i_1} & \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\,\Omega, \mathrm{Spec}\, L_1) \end{array} \tag{3}$$

We will show that this diagram commutes. Let $\psi^\sharp \in \mathrm{Aut}_k(L_2)$. Along the top, this is send to $\varphi_{L_1,L_2} \circ \psi \circ i_2$. Along the bottom, this is sent to $\psi|_{L_1} \circ i_1$. Here $\psi|_{L_1}$ denotes the automorphism of $\mathrm{Spec}\, L_1$ associated to $\psi^\sharp|_{L_1}$. Note that $i^\sharp_2 \circ \psi^\sharp \circ \varphi^\sharp_{L_1,L_2} = i^\sharp_1 \circ \psi^\sharp|_{L_1}$. We conclude that the diagram commutes. Observe that the group morphisms $|_{L_1} \colon \mathrm{Aut}_k(L_2) \to \mathrm{Aut}_k(L_1)$ for all $L_1, L_2 \in I$ such that $L_1 \leq L_2$ form a inverse system. We know that for all $L_1, L_2 \in I$ such that $L_1 \leq L_2$ and notation as in (3) the maps

$$(- \circ i_1)^{-1} \circ (\varphi_{L_1,L_2} \circ -) \circ (- \circ i_2) \colon \mathrm{Aut}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L_2) \to \mathrm{Aut}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L_1)$$

form an inverse system. The universal property of inverse limits gives an isomorphism of topological groups $\lambda : \varprojlim_{L \in I} \mathrm{Aut}_k(L) \to \varprojlim_{L\in I} \mathrm{Aut}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L)$. In the lemma below, we show that $\mathrm{Aut}_k(\Omega)$ can still be considered a topological group after changing to our unusual multiplication.

Lemma 3.18. *Let G be a topological group. Define G' to be the group given by G as a set and define multiplication by $a \cdot_{G'} b = b \cdot_G a$. Endow G' with the same topology as G. Then G' is a topological group.*

Proof. Since $G \to G$ given by sending a to a^{-1} is continuous, the map $G' \to G'$ given by sending a to a^{-1} is continuous. We have the following commutative diagram.

$$\begin{array}{ccc} G' \times G' & \longrightarrow & G' \\ \Big\downarrow & \nearrow & \\ G \times G & & \end{array}$$

The map $G' \times G' \to G'$ sends (a, b) to $a \cdot_{G'} b = b \cdot_G a$. The map $G' \times G' \to G \times G$ sends (a, b) to (b, a). The map $G \times G \to G$ sends (a, b) to $a \cdot_G b$. It follows that the map $G' \times G' \to G'$ is continuous, so G' is a topological group. □

Lemma 3.19. *The étale fundamental group of* $\mathrm{Spec}\, k$ *is isomorphic as topological groups to* $\mathrm{Aut}_k(\Omega)$, *where* $\psi^\sharp_1 \cdot \psi^\sharp_2 = \psi^\sharp_2 \circ \psi^\sharp_1$ *gives the multiplication on* $\mathrm{Aut}_k(\Omega)$.

Proof. We know that $\mathrm{Aut}_k(\Omega) \to \varprojlim_{L\in I} \mathrm{Aut}_k(L)$ sending ψ to $(\psi|_L)_{L\in I}$ is a homeomorphism [Mil22, Example 7.26]. It follows that it is an isomorphism under our unusual group structures. From the argument above, it follows that $\mathrm{Aut}_k(\Omega) \simeq \varprojlim_{L\in I} \mathrm{Aut}_k(L) \simeq \varprojlim_{L\in I} \mathrm{Aut}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L)$. □

Let Y be an étale cover of k. Lemma 3.19 gives a description of the étale fundamental group as a group. We will now show that this isormophism behaves well with respect to the group action of the étale fundamental group on $F(Y)$. Lemma 3.20 makes precise what behaving well with respect to the group action of the étale fundamental group of $F(Y)$ means. Asssume that Y is nonempty and consider the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L_1) & \longrightarrow & \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L_1, Y) \\ \big\downarrow \varphi^{\sharp}_{L_1,L_2}\circ - & & \big\downarrow -\circ\varphi_{L_1,L_2} \\ \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L_2) & \longrightarrow & \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L_2, Y) \end{array}$$

Note that the sets $\mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L)$ and maps $\varphi_{L_1,L_2}\circ -\colon \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L_1) \to \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L_2)$ are a direct system over I. The universal property of direct limits gives a bijection $\lambda'\colon \varinjlim_{L\in I} \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L) \to \varinjlim_{L\in I} \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L, Y)$. If $Y = \emptyset$, then $R = \{0\}$ so $\mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L) = \emptyset$. In this case $\lambda'\colon \varinjlim_{L\in I} \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L) \to \varinjlim_{L\in I} \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L, Y)$ is the unique bijection between emptysets.

Lemma 3.20. *The following diagram commutes. If Y is nonempty, the top horizontal map sends $((\psi_L)_{L\in I}, [g])$ to $[g \circ \psi_L^{-1}]$. The bottom horizontal map sends $((\psi_L^{\sharp})_{L\in I}, [g^{\sharp}])$ to $[(\psi_L^{\sharp})^{-1} \circ g^{\sharp}]$.*

$$\begin{array}{ccc} \varprojlim_{L\in I} \mathrm{Aut}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L) \times \varinjlim_{L\in I} \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L, Y) & \longrightarrow & \varinjlim_{L\in I} \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L, Y) \\ \big\downarrow \lambda\times\lambda' & & \big\downarrow \lambda' \\ \varprojlim_{L\in I} \mathrm{Aut}_k(L) \times \varinjlim_{L\in I} \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L) & \longrightarrow & \varinjlim_{L\in I} \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L) \end{array}$$

Proof. If Y is nonempty, one can verify this by a diagram chase. If Y is empty, all sets are the emptyset so the result follows. □

Observe that the map $\varprojlim_{L\in I} \mathrm{Aut}_k(L) \times \varinjlim_{L\in I} \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L) \to \varinjlim_{L\in I} \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L)$ from Lemma 3.20 defines a left-action of $\varprojlim_{L\in I} \mathrm{Aut}_k(L)$ on $\varinjlim_{L\in I} \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L)$ if Y is nonempty. If Y is empty, we obtain the trivial group action. By extending this action through the isomorphism $\mathrm{Aut}_k(\Omega) \simeq \varprojlim_{L\in I} \mathrm{Aut}_k(L)$, it follows that $\mathrm{Aut}_k(\Omega)$ acts on $\varinjlim_{L\in I} \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(R, L)$ in a way that is compatible with the group action of the étale fundamental group of $\mathrm{Spec}\, k$ on $\varinjlim_{L\in I} \mathrm{Hom}_{\mathrm{Spec}\, k}(\mathrm{Spec}\, L, Y)$. This finishes our discussion of the étale fundamental group of $\mathrm{Spec}\, k$.

3.3 Fiber functor in the case of a point

Notation 3.21. Let G be a group and S a G-set. The permutation group of S is denoted by $\mathrm{Perm}(S)$. The map from G to $\mathrm{Perm}(S)$ giving the group action is denoted by $\pi\colon G \to \mathrm{Perm}(S)$. The map from $G\times S$ to S giving the group action is denoted by $-\cdot -\colon G\times S \to S$.

Definition 3.22. *Let G be a topological group and S a G-set. Then S is called a discrete G-set if $-\cdot -\colon G \times S \to S$ is continuous. Here S has the discrete topology and $G \times S$ the product topology.*

From section 2.4 we know that $F(Y)$ is a finite $\pi_1(\mathrm{Spec}\, k, \overline{s})$-set. We show that $F(Y)$ is a discrete $\pi_1(\mathrm{Spec}\, k, \overline{s})$-set.

Lemma 3.23. *Let G be a group and S be a G-set. Assume G is also a topological space (not necessarily a topological group). Endow S and $\operatorname{Perm}(S)$ with the discrete topology and $G \times S$ with the product topology. If $\pi\colon G \to \operatorname{Perm}(S)$ is continuous, then $- \cdot - \colon G \times S \to S$ is continuous.*

Proof. Let $s \in S$, then

$$\begin{aligned} - \cdot -^{-1}(\{s\}) &= \{(g, s') \in G \times S | g \cdot s' = s\} \\ &= \bigcup_{\varphi \in \operatorname{Perm}(S)} \{(g, s') \in \pi^{-1}(\{\varphi\}) \times \{\varphi^{-1}(s)\}\}. \end{aligned}$$

Note that $\{\varphi^{-1}(s)\}$ is open and $\pi^{-1}(\{\varphi\})$ is open since π is continuous. We conclude that $- \cdot -^{-1}(\{s\})$ is open in the product topology, so $- \cdot -$ is continuous. ☐

Lemma 3.24. *Let $k_1, \ldots, k_n$ be finite separable extensions of k and denote $R = k_1 \times \ldots \times k_n$. The group morphism $\pi : \operatorname{Aut}_k(\Omega) \to \operatorname{Perm}(\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R, \Omega))$ given by sending $\psi^\sharp$ to $(\psi^\sharp)^{-1} \circ -$ is continuous. Here $\operatorname{Perm}(\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R, \Omega))$ carries the discrete topology.*

Proof. Since $\operatorname{Aut}_k(\Omega)$ is a topological group, it is sufficient to show that $\pi^{-1}(\mathrm{Id})$ is open. Fix embeddings $k_1 \subseteq \Omega, \ldots, k_n \subseteq \Omega$. Then $k_1 = k(a_{1,1}), \ldots, k_n = k(a_{n,1})$ for some $a_{1,1}, \ldots, a_{n,1} \in \Omega$. Denote by $a_{i,1}, \ldots, a_{i,n_i} \in \Omega$ all zeros of the minimal polynomial of $a_{i,1}$ over k for all $1 \leq i \leq n$. Consider the field $L = k(a_{1,1}, \ldots, a_{1,n_1}, \ldots, a_{n,1}, \ldots, a_{n,n_n})$. We show that $\pi^{-1}(\mathrm{Id}) = \operatorname{Aut}_L(\Omega)$. Let $\psi^\sharp \in \operatorname{Aut}_L(\Omega)$ and $g^\sharp \in \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R, \Omega)$. By Lemma 3.12, $\operatorname{Im}(g^\sharp) \subseteq L$ so $(\psi^\sharp)^{-1} \circ g^\sharp = g^\sharp$. This gives $\operatorname{Aut}_L(\Omega) \subseteq \pi^{-1}(\mathrm{Id})$. Let $\psi^\sharp \in \pi^{-1}(\mathrm{Id})$. For all $1 \leq i \leq n$ and $1 \leq j \leq n_i$, there exist $g^\sharp : R \to \Omega$ such that $\operatorname{Im}(g^\sharp) = k(a_{i,j})$. Since $(\psi^\sharp)^{-1} \circ g^\sharp = g^\sharp$ for all $g^\sharp \in \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R, \Omega)$, it follows that $\psi^\sharp(a_{i,j}) = a_{i,j}$. This gives $\psi^\sharp \in \operatorname{Aut}_L(\Omega)$. Since L is a finite extension of k, $\operatorname{Aut}_L(\Omega)$ is an open subset of $\operatorname{Aut}_k(\Omega)$ [Mil22, Theorem 7.13]. ☐

Proposition 3.25. *Let Y be an étale cover of $\operatorname{Spec} k$. Then $F(Y)$ is a finite, discrete $\pi_1(\operatorname{Spec} k, \overline{s})$-set.*

Proof. We already know that $F(Y)$ is a finite $\pi_1(\operatorname{Spec} k, \overline{s})$-set. Lemma 3.23 and Lemma 3.24 show that $\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R, \Omega)$ is a discrete $\operatorname{Aut}_k(\Omega)$-set. Now Proposition 3.15, Lemma 3.19 and Lemma 3.20 give the desired result. ☐

From Proposition 3.25, it follows that $F(Y)$ is actually a functor into the category of discrete, finite $\pi_1(\operatorname{Spec} k, \overline{s})$-sets.

Notation 3.26. Recall that we denote the category of discrete finite $\pi_1(\operatorname{Spec} k, \overline{s})$-sets by **DFPSet**.

We finish this section by showing that the fiber functor from the category of (finite) étale covers of $\operatorname{Spec} k$ to the category of discrete, finite $\pi_1(\operatorname{Spec} k, \overline{s})$-sets is an equivalence of categories.

Notation 3.27. Let S be a k-algebra, $g^\sharp\colon S \to \Omega$ be a k-algebra morphism and $k \subseteq L \subseteq \Omega$ a finite field extension of k such that $\operatorname{Im}(g^\sharp) \subseteq L$. Then $g^\sharp$ denotes both the morphism with image L and the map with image Ω.

Lemma 3.28. *The functor $F\colon \mathbf{F\acute{E}t}/\operatorname{Spec} k \to \mathbf{DFPSet}$ is fully faithful.*

Proof. Let Y_1, Y_2 be étale covers of $\operatorname{Spec} k$. If $Y_1 = \emptyset$, then $\operatorname{Hom}_{\operatorname{Spec} k}(Y_1, Y_2)$ contains one element and $\operatorname{Hom}_{\mathbf{DFPSet}}(F(Y_1), F(Y_2)) = \operatorname{Hom}_{\mathbf{DFPSet}}(\emptyset, F(Y_2))$ contains one element. The result follows. Now assume $Y_1 \neq \emptyset$.
From Lemma 3.6 we obtain bijections $\operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, Y_1) \to \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_1, \Omega)$ and $\operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, Y_2) \to \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_2, \Omega)$. Consider $\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_1, \Omega)$ and $\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_2, \Omega)$ as objects in **DFPSet**. The $\pi_1(\operatorname{Spec} k, \operatorname{Spec} \Omega)$-action is induced by the bijections mentioned above. These bijections induce the following bijection.

$$\begin{aligned}\operatorname{Hom}_{\mathbf{DFPSet}}(\operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, Y_1), \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, Y_2)) \to \\ \operatorname{Hom}_{\mathbf{DFPSet}}(\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_1, \Omega), \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_2, \Omega)).\end{aligned}$$

Consider the following map of sets

$$\begin{aligned}\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_2, R_1) \to \\ \operatorname{Hom}_{\operatorname{Spec} k}(Y_1, Y_2) \to \\ \operatorname{Hom}_{\mathbf{DFPSet}}(\operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, Y_1), \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, Y_2)) \to \\ \operatorname{Hom}_{\mathbf{DFPSet}}(\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_1, \Omega), \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_2, \Omega)).\end{aligned}$$

Here $\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_2, R_1) \to \operatorname{Hom}_{\operatorname{Spec} k}(Y_1, Y_2)$ sends $g^\sharp$ to g. The map $\operatorname{Hom}_{\operatorname{Spec} k}(Y_1, Y_2) \to \operatorname{Hom}_{\mathbf{DFPSet}}(\operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, Y_1), \operatorname{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, Y_2))$ is given by F. Let $\xi^\sharp \in \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_2, R_1)$. The map above first sends $\xi^\sharp$ to ξ. Secondly, ξ is send to $\xi \circ -$. Finally, $\xi \circ -$ is send to $- \circ \xi^\sharp$. Because of this F is fully faithful if and only if the map

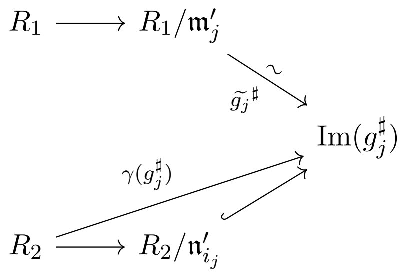
$$\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_2, R_1) \to \operatorname{Hom}_{\mathbf{DFPSet}}(\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_1, \Omega), \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_2, \Omega)),$$

given by sending $\xi^\sharp$ to $- \circ \xi^\sharp$ is a bijection. We first show that this map is injective. Let $\xi_1^\sharp, \xi_2^\sharp \in \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_2, R_1)$ such that $- \circ \xi_1^\sharp = - \circ \xi_2^\sharp$. Then $R_1/\mathfrak{m}'_j \simeq k_j$ as k-algebras. Choose an inclusion $k_j \hookrightarrow \Omega$ and consider the morphism of k-algebras $g_j^\sharp \colon R_1 \to R_1/\mathfrak{m}_j \hookrightarrow \Omega$. Since $g_j^\sharp \circ \xi_1^\sharp = g_j^\sharp \circ \xi_2^\sharp$ for all $1 \leq j \leq n_1$ it follows that $\xi_1^\sharp = \xi_2^\sharp$.

Next, we show that the map is surjective. Let $\gamma \in \operatorname{Hom}_{\mathbf{DFPSet}}(\operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_1, \Omega), \operatorname{Hom}_{k\text{-}\mathbf{Alg}}(R_2, \Omega))$. Let $g_j^\sharp$ be as before. By Lemma 3.12, $\gamma(g_j^\sharp)$ factors through a prime ideal $\mathfrak{n}'_{i_j}$ of R_2. It follows that $\operatorname{Im}(\gamma(g_j^\sharp)) \simeq k'_{i_j}$ as k-algebras. Lemma 3.9 implies that there exists a finite Galois extension of k, $k \subseteq L' \subseteq \Omega$ such that $\operatorname{Im}(g_j^\sharp), \operatorname{Im}(\gamma(g_j^\sharp)) \subseteq L'$. Let $\psi^\sharp \in \operatorname{Aut}_{\operatorname{Im}(g_j^\sharp)}(L')$, then there exists $(\psi_L^\sharp)_{L \in I} \in \varprojlim_{L \in I} \operatorname{Aut}_k(L)$ such that $\psi_{L'}^\sharp = \psi^\sharp$. Lemma 3.20 gives

$$\begin{aligned}&\gamma((\psi_L)_{L \in I} \cdot g_j^\sharp) = \gamma(((\psi_L)_{L \in I} \cdot g_j)^\sharp) = \gamma((\psi_{L'}^\sharp)^{-1} \circ g_j^\sharp) = \gamma((\psi^\sharp)^{-1} \circ g_j^\sharp) = \gamma(g_j^\sharp) \\ &(\psi_L)_{L \in I} \cdot \gamma(g_j^\sharp) = (\psi_{L'}^\sharp)^{-1} \circ \gamma(g_j^\sharp).\end{aligned}$$

Since γ is a morphism in the category **DFPSet**, $\gamma(g_j^\sharp) = \gamma((\psi_L)_{L \in I} \cdot g_j^\sharp) = (\psi_L)_{L \in I} \cdot \gamma(g_j^\sharp) = (\psi_{L'}^\sharp)^{-1} \circ \gamma(g_j^\sharp)$. It follows that $\operatorname{Aut}_{\operatorname{Im}(g_j^\sharp)}(L') \subseteq \operatorname{Aut}_{\operatorname{Im}(\gamma(g_j^\sharp))}(L')$. This gives $\operatorname{Im}(\gamma(g_j^\sharp)) \subseteq \operatorname{Im}(g_j^\sharp)$. We obtain the following commutative diagram.



This gives a morphism of k-algebras $(\widetilde{g_j}^\sharp)^{-1} \circ \gamma(g_j^\sharp) \colon R_2 \to R_1/\mathfrak{m}_j'$. Since $R_1/\mathfrak{m}_j' = k_j$, the universal property of the product gives a k-algebra morphism $\xi^\sharp \colon R_2 \to R_1$. We show that $\gamma = - \circ \xi^\sharp$. For all $1 \leq j \leq n_1$ and $z \in R_2$

$$\begin{aligned} g_j^\sharp \circ \xi^\sharp(z) &= g_j^\sharp((\widetilde{g_1}^\sharp)^{-1} \circ \gamma(g_1^\sharp)(z), \ldots, (\widetilde{g_{n_1}}^\sharp)^{-1} \circ \gamma(g_{n_1}^\sharp)(z))) \\ &= \widetilde{g_j}^\sharp((\widetilde{g_j}^\sharp)^{-1} \circ \gamma(g_j^\sharp)(z)) = \gamma(g_j^\sharp)(z). \end{aligned} \tag{4}$$

Let $g^\sharp : R_1 \to \Omega$ be a k-algebra morphism. Lemma 3.12 implies that $g^\sharp$ factors through $R_1/\mathfrak{m}_j'$ for some $1 \leq j \leq n_1$. Lemma 3.9 shows that exists a finite Galois extension $k \subseteq L' \subseteq \Omega$ such that $\mathrm{Im}(g^\sharp), \mathrm{Im}(g_j^\sharp) \subseteq L'$. Since $\mathrm{Im}(g^\sharp), \mathrm{Im}(g_j^\sharp)$ both factor through $R_1/\mathfrak{m}_j'$, there exists some $\psi^\sharp \in \mathrm{Aut}_k(L')$ such that $g^\sharp = (\psi^\sharp)^{-1} \circ g_j^\sharp$. Note that there exists $(\psi_L^\sharp)_{L \in I} \in \varprojlim_{L \in I} \mathrm{Aut}_k(L)$ such that $\psi_{L'}^\sharp = \psi^\sharp$. Now (4) and Lemma 3.20 give

$$\gamma(g^\sharp) = \gamma((\psi^\sharp)^{-1} \circ g_j^\sharp) = \gamma((\psi_L)_{L \in I} \cdot g_j^\sharp) = (\psi_L)_{L \in I} \cdot \gamma(g_j^\sharp) = (\psi^\sharp)^{-1} \circ \gamma(g_j^\sharp) = (\psi^\sharp)^{-1} \circ g_j^\sharp \circ \xi^\sharp = g^\sharp \circ \xi^\sharp.$$

This finishes the proof. □

The proof of the lemma below is based on the proof of [Zar15, Theorem 3.17].

Lemma 3.29. *The functor* $F \colon \mathbf{F\acute{E}t}/\operatorname{Spec} k \to \mathbf{DFPSet}$ *is essentially surjective.*

Proof. Let S be an object of $\mathbf{DFPSet}$. If S is empty, then $S \simeq F(\emptyset)$. Assume S is nonempty. Then $S = \bigsqcup_{i=1}^n S_i$, where S_i denote the orbits of the action of $\pi_1(\operatorname{Spec} k, \overline{s})$ on S. Through the isomorphism from Lemma 3.19, we consider S as a finite, discrete $\mathrm{Aut}_k(\Omega)$-set. Fix elements $s_1 \in S_1, \ldots, s_n \in S_n$. Consider some $1 \leq j \leq n$. The constant map $\mathrm{Aut}_k(\Omega) \to S$ sending $\psi^\sharp$ to s_j is continuous. Since the identity morphism is continuous, the universal property of the product gives a continuous map $\mathrm{Aut}_k(\Omega) \to \mathrm{Aut}_k(\Omega) \times S$ that sends $\psi^\sharp$ to $(\psi^\sharp, s_j)$. Since S is a discrete $\mathrm{Aut}_k(\Omega)$-set, we get a continuous map $\mathrm{Aut}_k(\Omega) \to \mathrm{Aut}_k(\Omega) \times S \to S$, sending $\psi^\sharp$ to $\psi^\sharp \cdot s_j$. Thus, the stabilizer $(\mathrm{Aut}_k(\Omega))_{s_j}$ of s_j is an open and closed subset, since S carries the discrete topology. Denote this stabilizer by H_j. Then [Mil22, Theorem 7.13] gives a field extension $k \subseteq \Omega^{H_j} \subseteq \Omega$. Since H_j is open and Ω is the separable closure, Ω^{H_j} is a finite, separable field extension of k, which we denote by k_j. Denote by $\widetilde{i_j}^\sharp \colon k_j \to \Omega$ the inclusion. Consider the étale cover $Y = \operatorname{Spec}(k_1 \times \ldots \times k_n)$ of $\operatorname{Spec} k$. We show that $F(Y) \simeq S$ in the category $\mathbf{DFPSet}$. The bijection $\mathrm{Hom}_{\operatorname{Spec} k}(\operatorname{Spec} \Omega, Y) \simeq \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(k_1 \times \ldots \times k_n, \Omega)$, endows the latter with the structure of a finite, discrete $\mathrm{Aut}_k(\Omega)$-set. Let $g^\sharp \in \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(k_1 \times \ldots \times k_n, \Omega)$. For any $1 \leq j \leq n$, denote by $i_j^\sharp \colon k_1 \times \ldots \times k_n \to \Omega$ the map $\widetilde{i_j}^\sharp \circ \pi_j^\sharp$. Here $\pi_j^\sharp : k_1 \times \ldots \times k_n \to k_1 \times \ldots \times k_n/\mathfrak{m}_j' = k_j$ denotes the projection map. By Lemma 3.12, $g^\sharp$ factors through k_j for some j. It follows that there exists $\psi^\sharp \in \mathrm{Aut}_k(\Omega)$ such that $\psi^\sharp \circ g^\sharp = i_j^\sharp$. Consider the map $\gamma \colon \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(k_1 \times \ldots \times k_n, \Omega) \to S$ given by sending $g^\sharp$ as above to $\psi^\sharp \cdot s_j$. We show that this map is well-defined. Let $\psi_2^\sharp \in \mathrm{Aut}_k(\Omega)$ such that $\psi_2^\sharp \circ g^\sharp = i_j^\sharp$. Then $(\psi^\sharp \circ (\psi_2^\sharp)^{-1}) \circ i_j^\sharp = i_j^\sharp$. It follows that $\psi^\sharp \circ (\psi_2^\sharp)^{-1} \in H_j$. This gives

$$\begin{aligned}(\psi^\sharp \circ (\psi_2^\sharp)^{-1}) \cdot s_j &= s_j \\ ((\psi_2^\sharp)^{-1} \cdot \psi^\sharp) \cdot s_j &= s_j \\ \psi^\sharp \cdot s_j &= \psi_2^\sharp \cdot s_j.\end{aligned}$$

We conclude that γ is well-defined. Now, we show that this map is a morphism of $\mathrm{Aut}_k(\Omega)$-sets. Let $g^\sharp \in \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(k_1 \times \ldots \times k_n, \Omega)$ be as above and consider some $\psi_3^\sharp \in \mathrm{Aut}_k(\Omega)$. Then $\psi^\sharp \circ \psi_3^\sharp \circ (\psi_3^\sharp)^{-1} \circ g^\sharp = i_j^\sharp$. From Lemma 3.20, it follows that

$$\gamma(\psi_3^\sharp \cdot g^\sharp) = (\psi^\sharp \circ \psi_3^\sharp) \cdot s_j = (\psi_3^\sharp \cdot \psi^\sharp) \cdot s_j = \psi_3^\sharp \cdot (\psi^\sharp \cdot s_j) = \psi_3^\sharp \cdot \gamma(g^\sharp).$$

Finally, we show that γ is a bijection. Let $z \in S$, then $z \in S_j$ for some j. Thus, there exits $\psi^\sharp \in \mathrm{Aut}_k(\Omega)$ such that $\psi^\sharp \cdot s_j = z$. It follows that $\gamma((\psi^\sharp)^{-1} \circ i_j^\sharp) = \psi^\sharp \cdot s_j = z$. Let $g_1^\sharp, g_2^\sharp \in \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(k_1 \times \ldots \times k_n, \Omega)$ such that $\gamma(g_1^\sharp) = \gamma(g_2^\sharp)$. Denote $\psi_1^\sharp, \psi_2^\sharp \in \mathrm{Aut}_k(\Omega)$ such that $\psi_1^\sharp \circ g_1^\sharp = i_{j_1}^\sharp, \psi_2^\sharp \circ g_2^\sharp = i_{j_2}^\sharp$. Since $\psi_1^\sharp \cdot s_{j_1} = \psi_2^\sharp \cdot s_{j_2}$, it follows that $j_1 = j_2$. Then $(\psi_1^\sharp)^{-1} \cdot \psi_2^\sharp \cdot s_{j_1} = s_{j_1}$. It follows that $(\psi_1^\sharp)^{-1} \cdot \psi_2^\sharp \in H_{j_1}$. This gives

$$\begin{aligned}((\psi_1^\sharp)^{-1} \cdot \psi_2^\sharp) \circ i_{j_1}^\sharp &= i_{j_1}^\sharp \\ (\psi_2^\sharp \circ (\psi_1^\sharp)^{-1}) \circ i_{j_1}^\sharp &= i_{j_1}^\sharp \\ (\psi_1^\sharp)^{-1} \circ i_{j_1}^\sharp &= (\psi_2^\sharp)^{-1} \circ i_{j_1}^\sharp \\ g_1^\sharp &= g_2^\sharp.\end{aligned}$$

We conclude that γ is bijective. Since S and $\mathrm{Hom}_{k\text{-}\mathbf{Alg}}(k_1 \times \ldots \times k_n, \Omega)$ both carry the discrete topology, it is a homeomorphism. This finishes the proof. □

Lemma 3.28 and Lemma 3.29 prove Theorem 2.42 in the case of $\operatorname{Spec} k$ as stated below.

Theorem 3.30. *The functor* $F\colon \mathbf{F\acute{E}t}/\operatorname{Spec} k \to \mathbf{DFPSet}$ *is an equivalence of categories.*

4 Elliptic curves

Throughout all this section, let k be an algebraically closed field of characteristic 0. By an *elliptic curve* over k we mean a smooth projective 1-dimensional variety E over k of genus 1, equipped with a geometric point $O \in E(k)$ which we call the point at infinity. We also write E/k to emphasize its base field. Elliptic curves have an underlying group structure, with the identity being the point O.

To calculate the étale fundamental group of an elliptic curve (E, O), let us first describe the category of finite étale covers $\mathbf{F\acute{E}t}/E$.

Let E/k be an elliptic curve and consider the map

$$\begin{aligned}[n] : E &\to E \\ P &\mapsto nP.\end{aligned}$$

This map maps P to $P + \ldots + P$ n-times, where n is an integer.

Definition 4.1. *[Sil09, page 66] An isogeny φ is a morphism*

$$\varphi : E_1 \to E_2$$

between elliptic curves E_1/k and E_2/k satisfying $\varphi(O_1) = O_2$, where O_1 and O_2 are the points at infinity of E_1 and E_2, respectively.

Example 4.2. The map $[n]$ is an isogeny.

This specific isogeny, the multiplication by n map, will be used repeatedly when computing the fundamental group of an elliptic curve. In order to do this, we need to introduce some results which will become useful later. We will present these results without proof, as the proofs are out of scope for this paper. However, the interested reader can find all of the proofs in [Kun17].

Proposition 4.3. *[Kun17, Proposition 4,14] Let E/k be an elliptic curve. Then the map $[n]$ is a finite étale map.*

We can use these maps to create an inverse system, by using the following propositions.

Proposition 4.4. *[Kun17, Proposition 4.13] Let $\varphi : X \to E$ be a finite étale cover where X is a k-scheme and E/k an elliptic curve. If X is connected, then X is also an elliptic curve.*

Lemma 4.5. *[Kun17, Lemma 5.11] Let $\varphi : X \to E$ be a finite étale cover where X/k and E/k are elliptic curves. Then (X, φ) is a Galois cover.*

These two propositions tell us that if we have a finite étale map $X \to E$ (as schemes) and E is an elliptic curve, then X is an elliptic curve and $\varphi : X \to E$ is a Galois cover.

In the following proposition we introduce the concept of *dual isogeny.*

Proposition 4.6. *[Sil09, page 86] Let $\varphi : E_1 \to E_2$ be an isogeny of elliptic curves, with $\deg \varphi = n$. Then there exists a unique isogeny $\hat{\varphi} : E_2 \to E_1$ such that*

$$\hat{\varphi} \circ \varphi = [n].$$

The unique isogeny described in Proposition 4.6 is called the dual isogeny.

Before we begin computing the étale fundamental group of an elliptic curve, let us define one more term.

Let $\{X_\beta\}_{\beta \in I}$ be a collection of Galois covers for E.

Definition 4.7. *[Kun17, Remark 5.9] A subcollection of Galois covers $\{X_\alpha\}_{\alpha \in J} \in \mathbf{F\acute{E}t}/E$ where $J \subset I$ is said to be cofinal if for any X_β, $\beta \in I$, there exists a cover Y_α, $\alpha \in J$ and a map $\varphi : X_\beta \to Y_\alpha$ making the following diagram*

$$\begin{array}{ccc} X_\beta & \xrightarrow{\varphi} & Y_\alpha \\ & \searrow \quad \swarrow & \\ & E & \end{array}$$

commute.

In this case we say that the map $X_\beta \to E$ is *dominated* by the map $Y_\alpha \to E$.

Example 4.8 (Computing the étale fundamental group of an elliptic curve)**.** [Kun17, Proposition 5.12] Let E/k an elliptic curve. In this example we compute the étale fundamental group of an elliptic curve in several steps.

Firstly, consider the multiplication by n map given at the beginning of this section. By Proposition 4.3, we know it is an étale map. Note that

$$\begin{aligned} [n] \circ [m] : E &\to E \\ P &\mapsto nmP \end{aligned}$$

thus $[n] \circ [m] = [nm]$. This allows us to define an inverse system, indexed by divisibility as in definition 2.36. In this sytem, we consider the pointed Galois covers to be (E, O) where O denotes the point at infinity. To see this, let the category be the category $\mathbf{F\acute{E}t}/E$, of finite étale covers over E and $\Lambda^{\mathrm{op}} = \mathbb{Z}_{>0}$ (according to definition 2.36). We have that for each $n, k \in \mathbb{Z}_{>0}$, with $n \leq k$ there are elliptic curves E_k, E_n such that there is a morphism

$$\begin{aligned} \varphi_{n,k} : E_k &\to E_n \\ P &\mapsto \frac{k}{n} P \end{aligned}$$

if $k = nm$ for some $m \in \mathbb{Z}_{>0}$. This morphism satisfies the identity property in definition 2.36, as

$$\varphi_{n,n}(P) = P.$$

Note that we have that $\varphi_{k,nk} = [n]$. Therefore, since in our direct system $E_i = E$ for all $i \in \mathbb{Z}_{>0}$ then for $n \leq m$ we have the following diagram

$$\begin{array}{ccc} E & \xrightarrow{[n]} & E \\ & \underset{[nm]}{\searrow} & \downarrow [m] \\ & & E \end{array}$$

which commutes because $[n] \circ [m] = [nm]$.

To see that the system is universal, consider an arbitrary connected finite étale cover $e \colon X \to E$. By Proposition 4.4, X is an elliptic curve, and e is an isogeny. Therefore by

Proposition 4.6, there exists $\hat{e}: E \to X$ such that $\hat{e} \circ e = [n]$. This states exactly that $(E, [n])$ dominates the cover $e: X \to E$, showing the universality of the system.

To compute the étale fundamental group, we consider the group of automorphisms of these étale covers, as

$$\pi_1^{\text{ét}}(E, O) = \varprojlim_n \text{Aut}(E, [n]).$$

Claim: The group of automorphisms $\text{Aut}(E, [n])$ is isomorphic to $\ker[n]$. The proof follows from [Kun17, Lemma 5.11] which in turn uses [Sil09, Sections 4, 6]. Consider the isogeny $[n] : E \to E$ and the map

$$\begin{aligned} \tau_P : E &\to E \\ Q &\mapsto Q + P \end{aligned}$$

where $P \in \ker[n]$. We have that their composition $[n] \circ \tau_P = [n]$ as

$$[n] \circ \tau_P(Q) = [n](P + Q) = nQ + nP = nQ$$

since $P \in \ker[n]$. Moreover, we can construct an inverse for τ_P, namely τ_{-P}. This implies $\tau_P \in \text{Aut}(E, [n])$. Consider the following homomorphism of groups

$$\begin{aligned} \xi : \ker[n] &\to \text{Aut}(E, [n]) \\ P &\to \tau_P. \end{aligned}$$

This homomorphism is injective as $\ker \xi = \{\mathcal{O}\}$ and therefore

$$\ker[n]/\ker \xi \cong \ker[n] \cong \xi(\ker[n])$$

so $\ker[n]$ is isomorphic to some subgroup of $\text{Aut}(E, [n])$.

To show ξ is surjective and thus $\ker[n] \cong \text{Aut}(E, [n])$, consider the map $\psi \in \text{Aut}(E, [n])$. Note that

$$\tau_{-\psi(O)} \circ \psi$$

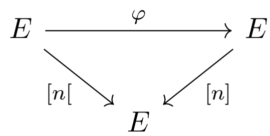
is an isogeny as $\tau_{-\psi(O)} \circ \psi(O) = \psi(O) - \psi(O) = O$. Therefore we can write this isogeny as φ, i.e.

$$\tau_{-\psi(O)} \circ \psi = \varphi.$$

Note that both ψ and $\tau_{-\psi(O)}$ are bijective as ψ is an automorphism and the map $\tau_{\psi(O)}$ is an inverse for $\tau_{-\psi(O)}$, thus φ is bijective. Therefore we can rewrite ψ as

$$\psi = \tau_{\psi(O)} \circ \varphi.$$

Moreover φ is a group homomorphism because it is an isogeny, hence it is an isomorphism, and specifically an isomorphism of covers implying the diagram



commutes. Thus,

$$[n] \circ \varphi = [n]$$

and we can see that for a point $Q \in E$ then $[n] \circ \varphi(Q) = n\varphi(Q) = nQ$ if and only if $\varphi(Q) = Q$ implying that φ is the identity. Thus we have that

$$\psi = \tau_{\psi(O)}$$

and hence ξ is surjective.

It follows from [Sil09, Corollary 6.4b] that

$$\ker[n] \cong (\mathbb{Z}/n\mathbb{Z})^2.$$

Putting it all together, we have that

$$\pi_1^{\text{ét}}(E, O) = \varprojlim_n \operatorname{Aut}(E, [n]) \cong \varprojlim_n ((\mathbb{Z}/n\mathbb{Z})^2, f)$$

where f is the quotient map

$$f : \mathbb{Z}/n\mathbb{Z} \to (\mathbb{Z}/n\mathbb{Z})/(\mathbb{Z}/k\mathbb{Z}) = (\mathbb{Z}/m\mathbb{Z})$$

for $n = mk$.

This is by definition the profinite completion $\hat{\mathbb{Z}}^2$ of $\mathbb{Z}^2$ hence

$$\pi_1^{\text{ét}}(E, O) \cong \hat{\mathbb{Z}}^2.$$

Remark 4.9. *It is worth noting that we computed the étale fundamental group of an elliptic curve in the case where the field has characteristic* 0*. For* $\operatorname{char} k = p > 0$ *with* p *prime, we have the following. Denote by* $\mathbb{Z}_p$ *the* p*-adic integers, then*

$$\pi_1^{\text{ét}}(E, O) \cong \prod_{l \neq p} \mathbb{Z}_l^2 \times \mathbb{Z}_p$$

where the product is taken over all primes $l \neq p$*. This result together with its proof can be found in [Kun17, Proposition 5.14].*

4.1 Comparison with the topological fundamental group

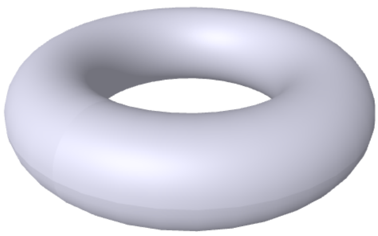
Once we compare the étale fundamental group with the topological fundamental group of an elliptic curve we get a quite nice comparison. This is described as follows.

Example 4.10. Consider an elliptic curve $E/\mathbb{C}$. It can be shown that with the Euclidean topology this curve is homeomorphic to a torus [Sil09, Proposition 3.6 b], see Figure 1a.

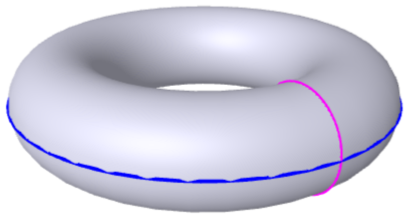
Intuitively, the fundamental group of the torus is generated by two loops, which are the pink and blue loops shown in Figure 1b. Both figures are generated using [The21].

Thus, the topological fundamental group of $E(\mathbb{C})$ centered at O under the Euclidean topology is isomorphic to the topological fundamental group of the torus which is $\mathbb{Z}^2$ [Arm04, page 100].

Therefore in a way, the étale and the topological fundamental group, in the case of elliptic curves, are comparable.



(a) Torus



(b) Generators of the topological fundamental group of the torus

5 Étale cohomology

As stated, the Zariski topology has too little open subsets, making the topological fundamental group not very interesting. The inadequacy of the Zariski topology becomes even more evident by Grothendiecks Vanishing Theorem: if X is an irreducible topological space, and $\mathcal{F}$ a constant sheaf, then

$$H^i(X, \mathcal{F}) = 0,$$

which make the cohomology of the constant sheaves not very interesting. This is (one of) the reasons one wishes to develop a more interesting cohomology theory for varieties.
This can be done in a very similar way as in the classical sense: we construct a category of sheaves for the étale topology on X which is an abelian category and has enough injectives, thus giving us a way to define étale cohomology as right derived functors.
We will work towards a link between étale cohomology and the étale fundamental group.

5.1 The étale site

Definition 5.1 (Site). *A site is a category $\mathcal{C}$ admitting fiber products, together with for each object U of $\mathcal{C}$ a set of families of maps $(U_i \to U)_{i \in i}$, called coverings of U, such that*

- *(identity axiom) for any U, the family (id_U) is a covering of U;*
- *(stability axiom) for any covering $(U \to U_i)_{i \in I}$ of U and any morphism $V \to U$, the familiy $(U_i \times_U V \to V)_{i \in I}$ is a covering of V;*
- *(transitivity axiom) if $(U_i \to U)_{i \in I}$ is a covering of U and if for each $i \in I$, the family $(V_{ij} \to U_i)_{j \in J_i}$ is a covering of U_i the family $(V_{ij} \to U)_{i,j}$ is a covering of U.*

We will often denote a site by its underlying category $\mathcal{C}$.

Definition 5.2 (Presheaf). *A presheaf of sets F (resp. groups, resp. rings) on a site $\mathcal{C}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to$ **Set** (resp. $F : \mathcal{C}^{\mathrm{op}} \to$ **Grp**, resp. $F : \mathcal{C}^{\mathrm{op}} \to$ **Ring**). A morphism of presheaves $\mathcal{F} \to \mathcal{G}$ is just a natural transformation.*

Definition 5.3 (Sheaf). *A sheaf of sets (resp. groups, resp. rings) is a presheaf of sets (resp. groups, resp. rings) such that it satisfies the sheaf condition: the sequence*

$$F(U) \longrightarrow \prod_{i \in I} F(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} F(U_i \times_U U_j)$$

is exact for every covering $(U_i \to U)_{i \in I}$. A morphism of sheaves $\mathcal{F} \to \mathcal{G}$ is a natural transformation.

Definition 5.4 (Surjective maps). *Let X be a variety or a scheme. A family of regular maps $(\varphi_i : U_i \to U)$ is called surjective if*

$$U = \bigcup_i \varphi_i(U_i)$$

Example 5.5. Let X be a topological space. We associate to X a category $\mathrm{Op}(X)$ with as objects the opens of X and a morphism $U \to V$ if and only if $U \subseteq V$. For any object U of $\mathrm{Op}(X)$ we declare $(U_i \to U)_{i \in I}$ covering if $\bigcup_i U = U$. This is a Grothendieck topology on $\mathrm{Op}(X)$ and a presheaf is a sheaf for this site if and only if it is a sheaf on the topological space X. This follows from the fact that $U_i \times_U U_j = U \cap V$.

Definition 5.6 (Small étale site)**.** *We define $X_{\text{ét}}$ to be the site with underlying category* $\text{Ét}/X$ *with objects étale morphisms $U \to X$ and arrows $V \to U$ making the triangle*

$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow & \downarrow \\ & & X \end{array}$$

commute. The covering families are the surjective families of étale morphisms. We call $X_{ét}$ the (small) étale site with $\text{Sh}(X_{\text{ét}})$ *its category of sheaves.*

Note that the étale site has as underlying category $\text{Ét}/X$, rather than Sch/X, which is often called the big étale site $X_{\text{Ét}}$ when endowed with the étale topology. This is because one can define the category of sheaves over $X_{\text{ét}}$ but not over $X_{\text{Ét}}$.

Proposition 5.7. *The category* $\text{Sh}(X_{\text{ét}})$ *is an abelian category.*

Proof. See [Mil13, Proposition 7.8]. □

Proposition 5.8. *The category* $\text{Sh}(X_{\text{ét}})$ *has enough injectives.*

Proof. See [Mil13, Proposition 8.12]. □

5.2 Étale cohomology

By the observation that $\text{Sh}(X_{\text{ét}})$ is an abelian category and has enough injectives, we can define étale cohomology similarly as how one defines sheaf cohomology in the classical sense.

Definition 5.9 (Cohomology)**.** *We define $H^i(X_{\text{ét}}, -)$ to be the i'th right derived functor of*

$$\text{Sh}(X_{\text{ét}}) \to \text{Ab} : \mathcal{F} \mapsto \Gamma(X, \mathcal{F}),$$

where Ab *is the category of abelian groups.*

This is well defined because the global sections functor is left exact.

Proposition 5.10. *We have the following properties of $H^i(X_{\text{ét}}, -)$:*

i) For any sheaf $\mathcal{F}$, $H^0(X_{\text{ét}}, \mathcal{F}) = \Gamma(X, \mathcal{F})$,

ii) If $\mathcal{I}$ is injective, then $H^i(X_{\text{ét}}, \mathcal{I}) = 0$ for each $i > 0$,

iii) A short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

gives a long exact sequence in a functorial manner

$$0 \to H^0(X_{\text{ét}}, \mathcal{F}') \to H^0(X_{\text{ét}}, \mathcal{F}) \to H^0(X_{\text{ét}}, \mathcal{F}'') \to H^1(X_{\text{ét}}, \mathcal{F}') \to \cdots$$

The properties *i-iii* listed in Proposition 5.10 determine the functors $H^i(X_{\text{ét}}, -)$ up to unique isomorphism.

5.3 Principal homogeneous spaces

Definition 5.11 (Principal homogeneous space)**.** *Let $\mathcal{G}$ be a sheaf of groups on $X_{\text{ét}}$ and $\mathcal{S}$ be a sheaf of sets on $X_{\text{ét}}$ on which $\mathcal{G}$ acts on the right i.e. a map $\mathcal{S} \times \mathcal{G} \to \mathcal{S}$ which defines a right action $\mathcal{G}(U) \times \mathcal{S}(U) \to \mathcal{S}(U) : (g, s) \mapsto sg$ for every object U in $X_{\text{ét}}$. Then $\mathcal{S}$ is called a principal homogeneous space for $\mathcal{G}$ if*

1. *there is an étale covering $(U_i \to X)_{i \in I}$ of X such that for all $i \in I$, $\mathcal{S}(U_i) \neq \emptyset$ in which case we say that the cover splits $\mathcal{S}$,*
2. *for every $U \to X$ étale and $s \in \Gamma(U, \mathcal{S}) = \mathcal{S}(U)$, the map $\mathcal{G}|_U \to \mathcal{S}|_U$, $g \mapsto sg$ is an isomorphism of sheaves.*

Definition 5.12 (Galois coverings)**.** *Let G be a finite group and $Y \to X$ a finite étale morphism such that G acts on Y on the right. Then $Y \to X$ is called a Galois cover with group G if the morphism*

$$\coprod_{g \in G} Y \to Y \times_X Y : (y, g) \mapsto (y, yg)$$

is an isomorphism.

Lemma 5.13. *Let G be a finite group and $\mathcal{G}$ the sheafification of the constant presheaf of G. Then we have a bijection*

$$\{\text{Galois coverings of } X \text{ with group } G\} \simeq \{\text{ principal homogeneous spaces for } \mathcal{G}\}$$

Proof. See [Mil13, Example 11.3] □

Theorem 5.14. *Let G be a finite group and $\mathcal{G}$ the sheafification of the constant presheaf of G. If X is connected there is a canonical isomorphism*

$$H^1(X_{\text{ét}}, \mathcal{G}) \simeq \operatorname{Hom}_{\text{cont}}(\pi_1(X, x), G),$$

where by $\operatorname{Hom}_{\text{cont}}$ we mean the continuous homomorphisms where G carries the discrete topology.

Proof. If $\check{H}^1$ denotes the first ech cohomology group and $\mathcal{U} = (U_i \to X)_i$ an étale covering of X, we have

$$\begin{aligned} H^1(X_{\text{ét}}, \mathcal{G}) &\simeq \check{H}^1(X_{\text{ét}}, \mathcal{G}) \\ &:= \varinjlim_{\mathcal{U}} \check{H}^1(X_{\text{ét}}, \mathcal{U}) \\ &\simeq \varinjlim_{\mathcal{U}} \{\text{principal homogeneous spaces for } \mathcal{G} \text{ split by } \mathcal{U}\} \\ &\simeq \{\text{principal homogeneous spaces for } \mathcal{G}\} \\ &\simeq \{\text{Galois coverings of } X \text{ with group } G\} \\ &\simeq \operatorname{Hom}_{\text{cont}}(\pi_1^{\text{ét}}(X, x), G). \end{aligned}$$

The first isomorphism follows from [Mil13, Theorem 10.2], the second [Mil13, Proposition 11.1] the fourth is Lemma 5.13 and the last is [Mil13, Example 11.3] □

Example 5.15. Let k be an algebraically closed field. Then by Lemma 3.19 we have

$$\pi_1^{\text{ét}}(\operatorname{Spec} k, \overline{x}) \simeq \operatorname{Gal}(k^{\text{sep}}/k) = 0$$

and thus we have by Theorem 5.14, since we have a unique homomorphism $0 \to G$ for every finite G, that the cohomology group $H^1(\operatorname{Spec} k, \mathcal{G}) = 0$. In particular, we can conclude immediately from the definition that there is no nontrivial étale cover of $\operatorname{Spec} k$.

Example 5.16. Let $k = \mathbb{R}$ be the real numbers. Then we have $\mathbb{R}^{\text{sep}} = \mathbb{C}$ with $[\mathbb{C} : \mathbb{R}] = 2$. It follows that

$$\pi_1^{\text{ét}}(\operatorname{Spec} \mathbb{R}, \overline{x}) \simeq \operatorname{Gal}(\mathbb{C}/\mathbb{R}) \simeq \mathbb{Z}/2\mathbb{Z}.$$

If we have a continuous homomorphism $\varphi : \mathbb{Z}/2\mathbb{Z} \to G$ for some finite group G, we need that $\varphi : 0 \mapsto 0_G$ and that $\varphi(1)^2 = 0_G$. Thus φ can be identified with an element $g \in G$ such that $g^2 = 0_G$. Therefore

$$H^1(\operatorname{Spec} \mathbb{R}_{\text{ét}}, \mathcal{G}) \simeq \{g \in G \mid g^2 = 0_G\}.$$

In particular, we have constant sheaves for which $H^1(X_{\text{ét}}, \mathcal{G})$ is nontrivial.

References

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[Arm04] M.A Armstrong. *Basic Topology*. Springer (India) Pvt. Limited, 2004. URL: https://books.google.nl/books?id=8GmCPwAACAAJ.

[FdJ23] Carel Faber and Robin de Jong. *Syllabus Algebraic Geometry 2*. Mastermath, spring 2023 edition, 2023.

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer, 1977. doi:10.1007/978-1-4757-3849-0.

[Hat03] A. Hatcher. *Algebraic topology*. 06 2003. doi:10.1017/S0013091503214620.

[Kun17] Arnab Kundu. The étale fundamental group of an elliptic curve, 2017. URL: https://math.uchicago.edu/~may/REU2017/REUPapers/Kundu.pdf.

[Len08] H. W. Lenstra. Galois theory for schemes, 2008. URL: https://websites.math.leidenuniv.nl/algebra/GSchemes.pdf.

[MA67] J. P. Murre and S. Anantharaman. Lectures on an introduction to Grothendieck's theory of the fundamental group, 1967.

[Mil13] James S. Milne. Lectures on étale cohomology (v2.21), 2013. URL: https://www.jmilne.org/math/CourseNotes/LEC.pdf.

[Mil22] James S. Milne. Fields and galois theory (v5.10), 2022. URL: https://www.jmilne.org/math/CourseNotes/FT.pdf.

[Sil09] Joseph H Silverman. *The Arithmetic of Elliptic Curves*. Graduate texts in mathematics. Springer, Dordrecht, 2009. URL: https://cds.cern.ch/record/1338326, doi:10.1007/978-0-387-09494-6.

[The21] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.3)*, 2021. https://www.sagemath.org.

[Zar15] Sergio Zarabara. Étale cohomology over spec(k), 2015. URL: https://algant.eu/documents/theses/zarabara.pdf.